**Stats 318** **Spring 2020**

# Homework 4

Due Friday, May 22nd

Your homework should be submitted via gradescope by **Friday, May 22nd** at the latest. Thanks!

1. Consider the simple stochastic process defined by

$$Y_t = \rho Y_{t-1} + \varepsilon_t, \quad t = 1, \dots L;$$

here $\rho = 0.5$ and the $\varepsilon_t$'s are i.i.d. standard normal.

   (a) Show that $\mathcal{L}(Y_t|Y_{-t}) = \mathcal{L}(Y_t|Y_{t-1}, Y_{t+1})$, where $\mathcal{L}(\cdot)$ is the *law* or *distribution* of its argument, and the notation $Y_{-t}$ means all the $Y$'s except $Y_t$.

   (b) Show that $\mathcal{L}(Y_t|Y_{t-1}, Y_{t+1}) = N(\mu_t, \gamma^2)$, and calculate $\mu_t$ and $\gamma$ [*Hint:* First compute $\mathcal{L}(Y_t, Y_{t+1}|Y_{t-1})$ (easy), and then use the standard conditioning formula for multivariate Gaussians to get $\mathcal{L}(Y_t|Y_{t-1}, Y_{t+1})$.]

   (c) Sample a sequence with $L = 12$ subject to

$$Y_1 > Y_2 > \ldots > Y_L; \tag{1}$$

   that is, we want to sample from the multivariate density

$$g(y) \propto f_Y(y) 1_{\{y_1 > y_2 > \ldots > y_L\}}, \tag{2}$$

   where $f_Y$ is the density of the multivariate normal $N(\mu, \Sigma)$ and where the constant of proportionality makes sure that $g$ integrates up to one.

   Record the number of samples drawn. Repeat this procedure 100 times and record the total number of samples drawn.

   (d) Explain how you would sample from $g$ using the Gibbs sampler and write some code for implementing this new strategy. Sample a sequence with $L = 12$ subject to the constraints. Record the number of samples drawn. Repeat this procedure 100 times and record the total number of samples drawn.

   (e) Which procedure is more efficient and why?

2. Implement the Swendsen-Wang and Wolff algorithm for simulating a $64 \times 64$ Ising model $\pi(x) \propto e^{-\beta E(x)}$, $E = -\sum_{v \sim v'} x_v x_{v'}$, near the critical temperature (a bit below) defined as $1/\beta_c = T_c = 2/\log(1 + \sqrt{2}) \approx 2.269$. Perform numerical simulations and report on your findings.