Stats 318 **Spring 2020**

Homework 3

Due Friday, May 15th

Your homework should be submitted via gradescope by **Friday, May 15th** at the latest. Thanks!

1. Show that a stopping time for the Markov Chain $\{X_t\}$ is a randomized stopping time. Give an example showing that the converse is not true.

2. Exercise 12.4, Chapter 12 of Levin, Peres and Wilmer.

3. Suppose we have a finite graph $G = (V, E)$ and an integer $q \geq 1$. A q-coloring of G assigns one of the q possible colors to each vertex so that no two adjacent sites have the same color. The problem of finding whether there exists a q-coloring is difficult. For instance, if G is a triangle, then there are no 2-colorings. If G is a planar graph (which means that edges do not cross each other), then the famous 4-color theorem states that there exist q-colorings with $q \geq 4$.

   (a) Assume that q-colorings of a graph G exist. Let S be the set of all q-colorings (fixed q) and let π be the uniform distribution on S. Describe a Metropolis algorithm to sample from π. (You will need to show that your chain is aperiodic.)

   (b) Implement this algorithm and test it. Report on your findings.

4. A natural generalization of the hard-core model is to allow for different "packing intensities"of 1's in the graph (see Section 3.3.4 in the textbook). This is done by introducing a parameter $\lambda > 0$ and changing the distribution to

$$\pi(x) = \begin{cases} \frac{\lambda^{1(x)}}{Z(\lambda)}, & x \text{ is feasible,} \\ 0, & \text{otherwise,} \end{cases}$$

   where $1(x)$ is the number of ones in the graph (the number of occupied sites).

   (a) Show that the conditional probability that a vertex is occupied given values at all other vertices is given by

$$\begin{cases} \frac{\lambda}{1+\lambda}, & \text{all neighbours of } v \text{ are free,} \\ 0, & \text{otherwise.} \end{cases}$$

   (b) Construct an MCMC for this generalized hard-core model.

   (c) *Open ended bonus problem.* What can you say about the mixing time of your MCMC? What are upper and lower bounds? [Hint: Please check Section 5.4.]