Stats 318 **Spring 2020**

Homework 1

Due Friday, April 17

Your homework should be submitted via gradescope by **Friday, April 17** at the latest. Thanks!

1. **A beautiful thing.** Write a routine to generate samples of the Markov chain over $S = \mathbb{R}^2$

$$X_{t+1} = A_{\theta_{t+1}} X_t + b_{\theta_{t+1}}$$

where the θ_t's are iid with probability

$$P(\theta = 1) = .01, \quad P(\theta = 2) = .07, \quad P(\theta = 3) = .07, \quad P(\theta = 4) = .85$$

and the matrices and vectors are

$$A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0.16 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0.2 & -0.26 \\ 0.23 & 0.22 \end{pmatrix} \quad A_3 = \begin{pmatrix} -0.15 & 0.28 \\ 0.26 & 0.24 \end{pmatrix} \quad A_4 = \begin{pmatrix} 0.85 & 0.04 \\ -0.04 & 0.85 \end{pmatrix}$$

and

$$b_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0 \\ 1.6 \end{pmatrix}, b_3 = \begin{pmatrix} 0 \\ .44 \end{pmatrix}, \quad b_4 = \begin{pmatrix} 0 \\ 1.6 \end{pmatrix}$$

Plot sample paths starting from $X_0 = (0, 0)$. What is this thing?

2. **Epidemiological model**. During Lecture 1, I discussed the possibility of turning the Kermack and McKendrick deterministic model into a stochastic model (a Markov chain). Implement the chain with the parameters $N = 500$, $\beta = 0.5$ and $1/\gamma = 5$ as to reproduce my simulations. Plot a few sample paths and comment on what you see.

3. Suppose (X_t) is a Markov chain on a finite state space S with transition matrix $P(x, y)$ and let T_A be the first time the chain will visit a subset $A \subset S$: $T_A = \min\{t \geq 0 : X_t \in A\}$. Put $\mu(x) = \mathbb{P}(T_A < \infty | X_0 = x)$.

   (a) Show that

$$\mu(x) = \begin{cases} 1 & \text{if } x \in A, \\ \sum_{y \in S} P(x, y)\mu(y) & \text{otherwise.} \end{cases}$$

   (b) Show that any nonnegative vector $q(x)$, $x \in S$ which solves these equations also obeys $q(x) \geq \mu(x)$ for all x.

4. Suppose a player enters a game with \$100. On each play of the game, she wins \$1 with probability p or loses \$1 with probability $1 - p$. The gambler continues betting until she has \$300 or until she is broke. What is the probability that she will ever quit the game? What is the probability that she quits the game as a winner?