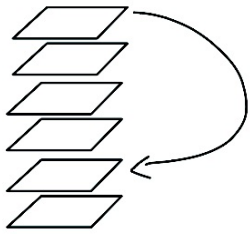
Stats 318: Lecture # 7

Agenda:

- Stationary times
- Strong stationary times
- Examples

Top-to-random shuffle

- Take top card and insert it at random location
- State space: all permutations S_n
- Equilibrium distribution: uniform over S_n



Q: How long does it take for this chain to mix?

A: About $n \log n$!

Clearly, original bottom card needs to reach the top

Why? Bottom cards still ordered (& prob bottom card at the top $= 0!$)

Stationary time

τ_{top}: Time at which bottom card reaches the top

Claim: $X_{\tau_{\text{top}}+1} \sim \pi$, chain is in equilibrium

Proof: At time t, say we have k cards under original bottom card

All $k!$ ordering are equally likely

Why? Induction on t,

- $t = 0$ ✓
- $t \geq 1$: if top card moved above ✓
  _____ below ✓

This proves the claim

Stationary time

Definition

$\{X_t\}$ irreducible chain with equil. dist. π. A stationary time τ is a randomized stopping time s.t. $\mathbb{P}_x(X_\tau = y) = \pi(y)$

Definition

$$X_0 = x \quad \& \quad X_{t+1} = f_{\theta_{t+1}}(X_t)$$

Randomized stopping time for $\{X_t\}$ is a stopping time for $\{\theta_t\}$

Example: $\tau = \tau_{\mathsf{top}}$ is a randomized stopping time

$\Longrightarrow \tau_{\mathsf{top}+1}$ stationary time for $\{X_t\}$

Example: Lazy RW on hypercube

- Recall equivalent description
  - Pick coordinate at random
  - Insert fair bit

$$\theta_t = \{i_t, \epsilon_t\}$$

coord. fair bit

- τ_{refresh}: first time all coord. have been selected
- τ_{refresh} is a stationary time
- Note that τ_{refresh} is not a stopping time for $\{X_t\}$

Strong stationary time

Observation: τ stationary time $\quad \not\Rightarrow \quad$ τ & X_τ independent

e.g. $Y \sim \pi$ & $X_0 = x_0$

$\tau = \min\{t \geq 0 : X_\tau = Y\}$ $\quad$ [Y random]

τ is stationary. Why?

τ is a randomized stopping time & $\mathbb{P}(X_\tau = y) = \mathbb{P}(Y = y) = \pi(y)$

However, $\tau = 0 \Longrightarrow X_\tau = x_0$; i.e. not indep.

e.g. same where chain is random walk on hypercube

$$\mathbb{P}(X_\tau = x \mid \tau = 1) \neq \mathbb{P}(X_\tau = x \mid \tau = 2)$$

Strong stationary time

Definition

Strong stationary time: τ stationary &

$$\mathbb{P}_x(\tau = t, X_\tau = y) = \mathbb{P}_x(\tau = t)\pi(y)$$

i.e. $X_\tau \sim \pi$ & is independent of τ

(1) $\tau_{\text{top}+1}$ strongly stationary

(2) τ_{refresh} ——————————

Lemma (Lemma 6.9)

Key Property

$$\mathbb{P}_x(\tau \leq t, X_t = y) = \mathbb{P}(\tau \leq t)\pi(y)$$

Distance to equilibrium

Proposition

If τ is a strong stationary time

$$d(t) = \max_x \ \|P^t(x,\cdot) - \pi\|_{\mathsf{TV}} \leq \max_x \ \mathbb{P}_x(\tau > t)$$

1) Separation distribution s_x

$$s_x(t) = \max_y \ \left[1 - \frac{P^t(x,y)}{\pi(y)}\right] \leq \mathbb{P}_x(\tau > t)$$

Why? $1 - \dfrac{P^t(x,y)}{\pi(y)} = 1 - \dfrac{\mathbb{P}_x(X_t = y)}{\pi(y)} \leq 1 - \dfrac{\mathbb{P}(X_t = y, \tau \leq t)}{\pi(y)} = \mathbb{P}_x(\tau > t)$

2) $\|P^t(x,\cdot) - \pi\|_{\mathsf{TV}} \leq s_x(t) \Longrightarrow d(t) \leq s(t) = \max_x s_x(t)$

Why? $\|P^t(x,\cdot) - \pi\|_{\mathsf{TV}} = \sum_{y:\pi(y)>P^t(x,y)} [\pi(y) - P^t(x,y)]$

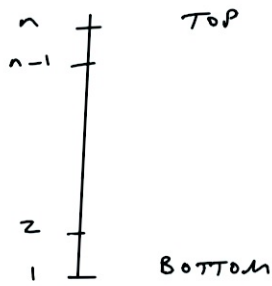
$$= \sum_{y:\pi(y)>P^t(x,y)} \pi(y) \left[1 - \frac{P^t(x,y)}{\pi(y)}\right] \leq s_x(t)$$

Example: Top-to-random shuffle

X_t: Location of bottom card

$P(k, k+1) = \frac{k}{n} \qquad k < n$

$P(k, k) = 1 - \frac{k}{n}$



Waiting time τ_k between k & $k+1$

$$\mathbb{E}\tau_k = \frac{n}{k}$$

Waiting time before reaching the top

$$\tau_{\text{top}} = \tau_1 + \tau_2 + \ldots + \tau_{n-1} \Longrightarrow \mathbb{E}\tau_{\text{top}} = n \sum_{k=1}^{n-1} \frac{1}{k} \approx n \log n$$

$\therefore$ Mixing time is roughly $n \log n$

Analogy with coupon collection

- X_t : # of distinct coupons at time t $(X_0 = 0)$
- $\begin{cases} P(k, k+1) = \frac{n-k}{n} & k < n \\ P(k, k) = \frac{k}{n} & \text{——} \end{cases}$
- Time it takes to complete the collection

$$\mathbb{E}\tau = \sum_{k=0}^{n-1} \frac{n}{n-k} \approx n \log n$$

Random walk on hypercube $\{0,1\}^n$

- Lazy version (stay put wp $\frac{1}{2}$)
- $\tau = \tau_{\text{refresh}}$

$$\mathbb{P}(\tau > n \log n + cn) \leq e^{-c}$$

- Why? Coupon collection
  - m draws
  - n items
  - A_i {item i not collected}

$$\mathbb{P}(\cup A_i) = \mathbb{P}(\tau > m) \leq \sum_i \mathbb{P}(A_i) = n\left(1 - \frac{1}{n}\right)^m \leq ne^{-m/n} = e^{-c}$$

last equality holds if $m = n \log n + cn$

- Mixing time at most $n \log n$ [off by a factor of 2]