Stats 318: Lecture # 6

Agenda:

- Mixing times
- Coupling & bounds on mixing times
- Examples

Mixing times

$$d(t) = \max_x \ \|P^t(x,\cdot) - \pi\|_{\mathsf{TV}}$$

Definition (Mixing time)

$$t_{\mathsf{mix}}(\epsilon) = \min\{t : d(t) \le \epsilon\}$$
$$t_{\mathsf{mix}} \quad = t_{\mathsf{mix}}(1/4)$$

Proposition

$$d(t) \le \overline{d}(t) \le 2d(t) \qquad \overline{d}(t) = \max_{x,y} \ \|P^t(x,\cdot) - P^t(y,\cdot)\|_{\mathsf{TV}}$$

Proof: $\overline{d}(t) \le 2d(t)$ follows from triangle inequality

To show $d(t) \leq \overline{d}(t)$ [Section 4.4 in textbook]

$$
\begin{aligned}
\|P^t(x,\cdot) - \pi\|_{\mathsf{TV}} &= \max_A |P^t(x,A) - \pi(A)| \\
&= \max_A \Big| \sum_y \pi(y)(P^t(x,A) - P^t(y,A)) \Big| \\
&\leq \sum_y \pi(y) \max_A \big| P^t(x,A) - P^t(y,A) \big| \\
&= \sum_y \pi(y) \|P^t(x,\cdot) - P^t(y,\cdot)\|_{\mathsf{TV}} \\
&\leq \max_y \|P^t(x,\cdot) - P^t(y,\cdot)\|_{\mathsf{TV}}
\end{aligned}
$$

Submultiplicativity

Proposition

$\overline{d}$ is submultiplicative : $\overline{d}(s+t) \leq \overline{d}(t)\overline{d}(s)$

X_s & Y_s optimal coupling so that

$$\|P^s(x,\cdot) - P^s(y,\cdot)\|_{\mathsf{TV}} = \mathbb{P}(X_s \neq Y_s) \qquad X_s \sim P^s(x,\cdot), Y_s \sim P^s(y,\cdot)$$

$$P^{s+t}(x,w) = \sum_z P^s(x,z)P^t(z,w) = \sum_z \mathbb{P}(X_s = z)P^t(z,w) = \mathbb{E}\left[P^t(X_s,w)\right]$$

$$\therefore \quad P^{s+t}(x,w) - P^{s+t}(y,w) = \mathbb{E}\left[P^t(X_s,w) - P^t(Y_s,w)\right]$$

$$\begin{aligned}\|P^{s+t}(x,\cdot) - P^{s+t}(y,\cdot)\|_{\mathsf{TV}} &= \tfrac{1}{2}\sum_w \left|\mathbb{E}\left[P^t(X_s,w) - P^t(Y_s,w)\right]\right| \\ &\le \tfrac{1}{2}\sum_w \mathbb{E}|P^t(X_s,w) - P^t(Y_s,w)| \\ &= \mathbb{E}\|P^t(X_s,w) - P^t(Y_s,w)\|_{\mathsf{TV}}\end{aligned}$$

$$\begin{aligned}\|P^t(X_s,\cdot) - P^t(Y_s,\cdot)\|_{\mathsf{TV}} &= 0 && \text{if } X_s = Y_s \\ &\le \overline{d}(t) && \text{always}\end{aligned}$$

$$\therefore \qquad \|P^{s+t}(x,\cdot) - P^{s+t}(y,\cdot)\|_{\mathsf{TV}} \le \overline{d}(t)\mathbb{P}(X_s \neq Y_s) = \overline{d}(t)\overline{d}(s)$$

Consequence: $k \in \mathbb{N}$

$$\begin{aligned}&d(kt) \le \overline{d}(kt) \le \left[\overline{d}(t)\right]^k \le [2d(t)]^k \\ &d(kt_{\mathsf{mix}}) \le 2^{-k} \qquad \Longrightarrow \qquad t_{\mathsf{mix}}(\epsilon) \le \left\lceil \log_2 \epsilon^{-1} \right\rceil t_{\mathsf{mix}}\end{aligned}$$

Further comments about mixing times

- Chain $\{X_t\}$ starts at x
- _____ $\{Y_t\}$ _____ y
- τ_{couple}: first time they meet

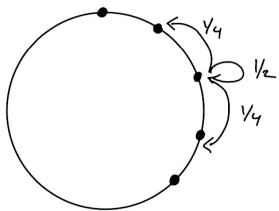
$$d(t) \leq \max_{x,y} \ \mathbb{P}_{x,y}(\tau_{\text{couple}} > t)$$

Because $\overline{d}(t) = \max_{x,y} \|P^t(x,\cdot) - P^t(y,\cdot)\|_{\text{TV}} \leq \max_{x,y} \ \mathbb{P}(\tau_{\text{couple}} > t)$

Why? See proof of second Ergodic theorem

Example 1: RW on cycle

$$\mathcal{X} = \mathbb{Z}/n\mathbb{Z} = \{0, 1, 2, \ldots, n-1\}$$



$$X_t = X_{t-1} + Z_t \qquad Z_t = \begin{cases} -1 & \text{wp } \frac{1}{4} \\ 0 & \text{wp } \frac{1}{2} \\ 1 & \text{wp } \frac{1}{4} \end{cases}$$

Coupling: $X_t, Y_t \qquad X_0 = x, Y_0 = y$

At each time, toss fair coin

$'H'$ $\begin{cases} X_{t+1} = X_t \pm 1 & \pm \text{ wp } \frac{1}{2} \text{ [Indep.]} \\ Y_{t+1} = Y_t \end{cases}$

$'T'$ $\begin{cases} X_{t+1} = X_t \\ Y_{t+1} = Y_t \pm 1 \end{cases}$ _____

$$X_t \sim P^t(x, \cdot)$$
$$Y_t \sim P^t(y, \cdot)$$

When do they meet?

- D_t, clockwise distance between X_t & Y_t
- D_t: SRW on $\{0, 1, 2, \ldots, n\}$
- $k = d(x, y)$

$$\mathbb{E}_{x,y}\{\tau_{\text{couple}}\} = \mathbb{E}_k\{\text{min } t : D_t \in \{0, n\}\} = k(n-k)$$

$$\therefore \quad d(t) = \max_{x,y} \mathbb{P}_{x,y}(\tau_{\text{couple}} > t) \leq \frac{n^2}{4t}$$

$$\therefore \quad t_{\text{mix}} \leq n^2$$

On the other hand, $t_{\text{mix}} \geq c \cdot n^2$ for some $c > 0$ [Why?]

Example 2: RW on Hypercube (Ehrenfest)

- Avoid periodicity via lazy chain on $\{0,1\}^n$

$$P(x,y) = \begin{cases} \frac{1}{2} & x = y \\ \frac{1}{2n} & x \sim y \end{cases}$$

- Equivalent description
  - Pick coord. at random
  - Place random fair bit
- Coupling: Two RW X & Y starting at x & y
  - Pick coord. at random
  - Replace coord. with same random fair bit
  - Walks will agree on selected coordinates

- $X_t \sim P^t(x, \cdot)$, $Y_t \sim P^t(y, \cdot)$
- τ_{couple}: First time all coordinates have been selected [coupon collection]

$$\mathbb{P}(\tau_{\mathsf{couple}} > n \log n + cn) \leq e^{-c}$$

$$\therefore \qquad d(n \log n + cn) \leq e^{-c}$$

$$\therefore \qquad t_{\mathsf{mix}}(\epsilon) \leq n \log n + (\log 1/\epsilon)n$$

Correct mixing time $\approx \frac{1}{2} n \log n$