Stats 318: Lecture # 5

Agenda:

- The entropy
- Kullback-Leibler distance
- Entropy is increasing
- Convergence of Markov chains (revisited)

Kullback-Leibler distance

π reference prob. dist. s.t. $\pi(x) > 0 \; \forall \; x \in \mathcal{X}$

Definition

Relative entropy of Kullback-Leibler divergence of prob. measure μ wrt π is

$$D(\mu \mid\mid \pi) = \sum_{x \in \mathcal{X}} \mu(x) \log \frac{\mu(x)}{\pi(x)}$$

Example: $|\mathcal{X}| = n$ and π uniform, $\pi(x) = 1/n$

$$D(\mu \mid\mid \pi) = \sum_x \mu(x) \log \mu(x) + \sum_x \mu(x) \log n = \log n - S(\mu)$$

Entropy used in ↑ statistical physics, information theory

KL distance

$D(\mu \mid\mid \pi)$ is a distance measure from μ to π

Proposition

- $D(\mu \mid\mid \pi) \geq 0 \quad \& \quad D(\mu \mid\mid \pi) = 0 \quad$ iff $\mu = \pi$
- $\mu \longmapsto D(\mu \mid\mid \pi)$ *is convex*

Proof: Convexity is immediate (do it!)

$$D(\mu \mid\mid \pi) = \sum_x \pi(x) \frac{\mu(x)}{\pi(x)} \log \frac{\mu(x)}{\pi(x)} = \mathbb{E}h(Y) \qquad Y = \frac{\mu(X)}{\pi(X)},\ X \sim \pi$$

$$h(t) = t \log t$$

$$\therefore \quad D(\mu \mid\mid \pi) \geq h(\mathbb{E}Y) = h(1) = 0 \text{ (Jensen's ineq.)} \qquad \mathbb{E}Y = 1$$

$$h \text{ strictly convex} \implies \text{equality iff } Y = \text{Cst}$$

$$\iff \mu(x) = \pi(x) \text{ for all } x \in \mathcal{X}$$

Second law of thermodynamics

Famous principle in physics: entropy is increasing (S. Carnot)

Example: Physical system with states $x \in \mathcal{X}$

$$S = -\sum_x p(x) \log p(x)$$

S increases with time; e.g. in Ehrenfest model

Where does this universal law come from?

Growth of entropy

X chain with trans. mat. P & equilibrium dist π
Then relative entropy of $\pi^{(t)}$ decreases as time goes by

Theorem

P irreducible & π invariant for all dist. μ

$$D(\mu P \mid\mid \pi) \leq D(\mu \mid\mid \pi)$$

If $\exists\ y$ with $P(x,y) > 0 \quad \forall\ x \in \mathcal{X}$, equality iff $\pi = \mu$

Implication: $X_t \sim \pi^{(t)} = \pi^{(0)} P^t$

$$D(\pi^{(0)}||\pi) \geq D(\pi^{(1)}||\pi) \geq \ldots \geq D(\pi^{(t)}||\pi) \geq \ldots$$

π uniform prob on $\mathcal{X} \Longrightarrow$ entropy is increasing with time $\rightsquigarrow$ '2^{nd} law becomes a theorem ...'

Proof of theorem

$$D(\mu P \mid\mid \pi) = \sum_x \pi(x) h\left(\frac{(\mu P)(x)}{\pi(x)}\right)$$

Claim: $$h\left(\frac{(\mu P)(x)}{\pi(x)}\right) \leq \sum_y \underbrace{\frac{\pi(y)}{\pi(x)} P(y,x)}_{q(x,y)} h\left(\frac{\mu(y)}{\pi(y)}\right) \qquad \sum_y q(x,y) = 1$$

With claim:
$$D(\mu P \mid\mid \pi) = \sum_x \pi(x) h\left(\frac{(\mu P)(x)}{\pi(x)}\right) \leq \sum_{x,y} \pi(y) P(y,x) h\left(\frac{\mu(y)}{\pi(y)}\right)$$
$$= \sum_y \pi(y) h\left(\frac{\mu(y)}{\pi(y)}\right) = D(\mu \mid\mid \pi)$$

Equality iff $\forall\ x$

$$h\left(\sum_y q(x,y)\frac{\mu(y)}{\pi(y)}\right) = \sum_y q(x,y) h\left(\frac{\mu(y)}{\pi(y)}\right)$$

By assumption $\exists\ x$ s.t. $P(y,x) > 0\ \forall\ y \qquad \Longrightarrow \qquad q(x,y) > 0$

h strictly cvx $\Longrightarrow \frac{\mu(y)}{\pi(y)} =$ Cst. for equality $\quad \therefore\ \mu = \pi$

Proof of claim: $\qquad \frac{\mu P(x)}{\pi(x)} = \sum_y \frac{\mu(y)P(y,x)}{\pi(x)} = \sum_y \frac{\mu(y)}{\pi(y)} q(x,y)$

By convexity, $\qquad \text{LHS} = h\left(\sum_y \frac{\mu(y)}{\pi(y)} q(x,y)\right) \leq \sum q(x,y) h\left(\frac{\mu(y)}{\pi(y)}\right)$

Monotonicity of (relative) entropy $\Longrightarrow$ chain converges

P strongly irreducible: suppose $P(x, y) > 0 \quad \forall\ x, y$ (otherwise work with P^k)

- $D(\pi^{(t)} \mid\mid \pi)$ nonincreasing + nonnegative $\Longrightarrow$ converges

$$D^* = \lim\ D(\pi^{(t)} \mid\mid \pi) = \lim\ D(\pi^{(t)} P \mid\mid \pi)$$

- Extract convergent subsequence $\pi^{(t_k)} \longrightarrow \pi^*$
- By continuity of $\mu \longmapsto D(\mu \mid\mid \pi)$

$$D^* = D(\pi^* \mid\mid \pi) = D(\pi^* P \mid\mid \pi) \quad \therefore \quad \pi^* = \pi$$

- Now $D(\pi^{(t)} \mid\mid \pi) \longrightarrow 0$ and whole sequence converges, i.e.

$$\pi^{(t)} \longrightarrow \pi \text{ as } t \longrightarrow \infty$$