Stats 318: Lecture 2

Agenda:

- Epidemiology Example
- MC and Random Recurrent Sequences
- Stationary Distributions
- Reversibility
- Existence of Stationary Distributions

Markov Chains and Random Recurrent Sequences

Random Recurrent Sequence

$$X_{t+1} = f(X_t, \theta_{t+1})$$

(i) $\theta_1, \theta_2, \ldots$, iid RVs taking values in Θ

(ii) Map $f : \Theta \times \mathcal{X} \to \mathcal{X}$ is measurable

(iii) X_0 is a RV (possibly deterministic) independent of $\{\theta_t\}_{t \geq 1}$

If f does not depend on θ, recurrence is deterministic

- Dynamical Systems Theory if X_0 deterministic
- Ergodic Theory if X_0 random

Canonical Representation

Claim

(i) A random dynamical system is a Markov chain

(ii) Conversely, every Markov chain with transition matrix P can be expressed as a random dynamical system

Proof

(i)

$$\begin{aligned}\mathbb{P}(X_{t+1} = x_{t+1} | X_{0:t} = x_{0:t}) = \mathbb{P}(f(x_t, \theta_{t+1}) = x_{t+1} | X_{0:t} = x_{0:t}) \\ = \mathbb{P}(f(\theta_{t+1}, x_t) = x_{t+1}) = P(x_t, x_{t+1})\end{aligned}$$

Proof

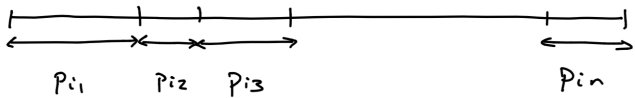
(ii) Finite/countable state space: e.g. $\mathcal{X} = \{1, \ldots, n\}$

For $i, j \in \mathcal{X}$ set $\quad \tau(i,j) = p_{i1} + \ldots + p_{ij} \quad \tau(i,0) = 0$

Now $\theta \sim U[0,1]$ and define $f : [0,1] \times \mathcal{X} \to \mathcal{X}$ by

$$f(\theta, i) = j \quad \text{if} \quad \tau(i, j-1) \leq \theta < \tau(i,j)$$

$$P(X_{t+1} = j | X_t = i) = p_{i,j} = P(\theta_{t+1} \in [\tau(i,j-1), \tau(i,j)$$



Kolmogorov Chapman Equations

- Initial distribution: $X_0 \sim \pi^{(0)}$
- Vector matrix product

$$(\pi P)(x) = \sum_y \pi(y) P(y, x)$$

- Interpretation

$$X_0 \sim \pi^{(0)} \qquad \Longrightarrow \qquad X_1 \sim \pi^{(1)} = \pi^{(0)} P$$

- P^n: n-th power of matrix P

$$P^{n+1}(y, x) = (P \cdot P^n)(y, x) = \sum_z P(y, z) P^n(z, x)$$

Kolmogorov Chapman Equations

Theorem

$\{X_t\}$ *is a MC with state space* $\mathcal{X}$

(i) $X_t \sim \pi^{(t)}$ *obeying recurrence relation*

$$\pi^{(t)} = \pi^{(t-1)} P = \pi^{(0)} P^t$$

(ii) *for all* $x, y \in \mathcal{X}$

$$P(X_t = x | X_0 = y) = P^t(y, x)$$

(iii) *For all bounded functions* $h : \mathcal{X} \to \mathbb{R}$

$$\mathbb{E}[h(X_t)|X_0 = y] = (P^t h)(y)$$

Proof

(i) $\pi^{(t+1)}(x) = P(X_{t+1} = m)$

$$= \sum_y P(X_{t+1} = m | X_t = y) P(X_t = y)$$

$$= \sum_y \pi^{(t)}(y) P(y, x)$$

$$= (\pi^{(t)} P)(x)$$

(ii) Apply (i) with $\pi^{(0)} = \delta(y)$

(iii)

$$\mathbb{E}[h(X_t)] = \sum_x h(x) [\pi^{(0)} P^{(t)}](x)$$

with $\pi^{(0)} = \delta(y)$, we have

$$\mathbb{E}[h(X_t) | X_0 = y] = \sum_x h(x) P^t(y, x) = (P^t h)(x)$$

Example

- Wish to transmit an n-bit signal through a channel
- Each bit is transmitted with an error probability

  $0 \to 1$ w.p. a $\quad 0 < a, b < 1$

  $1 \to 0$ – b
- Transmitted signal after t steps is X_t

  $\{0,1,0,0,\ldots,0\} \to \{0,1,1,0,\ldots,0\} \to \ldots \to$

  $X_0 \qquad\qquad X_1 \qquad\qquad X_t$
- Relays operate independently of each other
- Bit errors are independent

Q: Critical number of relays beyond which probability of receiving an incorrect message is above a tolerance level ε?

Suppose first $n = 1$
Sequence X_t has transition matrix

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$$

Put $g_t = P(X_t = 0)$. From theorem,

$$g_{t+1} = (1-a)g_t + b(1-g_t)$$

Fixed point $g = \frac{b}{a+b}$

$$g_t - g = (1-a-b)^t(g_0 - g)$$

Probability received message is correct:

$$r_t(0) = \frac{b}{a+b} + \frac{a}{a+b}(1-a-b)^t \quad X_0 = 0$$

$$r_t(1) = \frac{a}{a+b} + \frac{b}{a+b}(1-a-b)^t \quad X_0 = 1$$

Length message $n \geq 1$

$X_t = (X_t^1, \ldots, X_t^n)$ where variables are independent and have the same distribution for the same initial value of the bit

$$r_t = \prod_{i=1}^{n} r_t(X_0^i) \geq [\alpha + (1-\alpha)(1-a-b)^t]^n \qquad \alpha = min(g, 1-g)$$

$$RHS$$

choose t s.t. $\qquad 1 - RHS \leq \varepsilon$

Stationary Distributions/Equilibrium Measures

Definition

Let $\{X_t\}_{t\geq 0}$ be a Markov Chain with transition matrix P. A probability distribution π is said to be invariant or stationary if

$$\pi P = P$$

Interpretation: If initial distribution $\pi^{(0)} = \pi$, then

$$\pi^{(1)} = \pi P = \pi$$

and by integrating $\pi^{(t)} = \pi \qquad \forall\ t \geq 1$

From statistical viewpoint, system is in equilibrium
Other name: Equilibrium Distribution or Measure