Stats 318: Lecture # 17

Agenda: Countable Markov Chains

- Branching process
- Recurrence and transience
- Classification of states
- Limit theorems

Examples of countable state space

Symmetric random walk

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Branching processes

Originally considered by Galton & Watson in the 1870's while seeking for an explanation for the phenomenon of the disappearance of family names even in a growing population

Branching processes

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What is the probability that after n generations, an individual has no male descendants?

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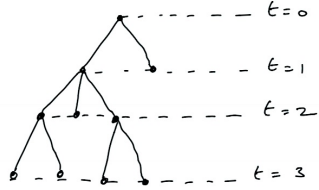
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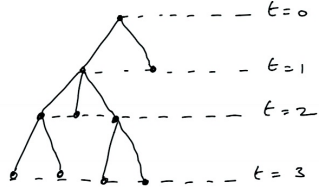
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$$X_{t+1} = \sum_{k=1}^{X_t} N_k^{(t+1)}$$

$\{X_t\}$ is a Markov chain



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$$\therefore \quad q_t = \varphi(q_{t-1}) \qquad \varphi(s) = \sum_{k \geq 0} p_k s^k \quad \text{(MGF of } N\text{)}$$
$$= \mathbb{E}(s^N)$$

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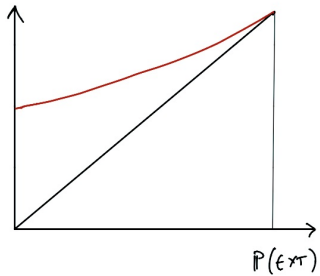
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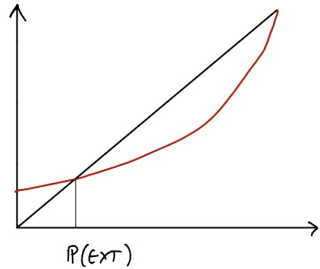
Remark: $\varphi(0) = p_0 = \mathbb{P}(N = 0)$ $\qquad$ $\varphi'(1) = \mathbb{E}\ N$

$\varphi(1) = 1$ $\qquad$ $\varphi'' \geq 0 \Longrightarrow \varphi$ is convex

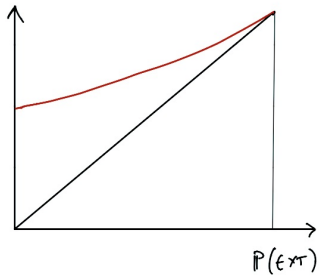
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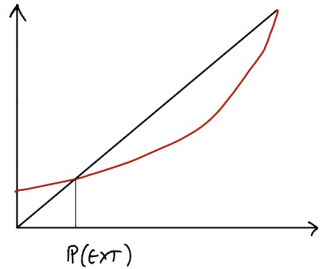
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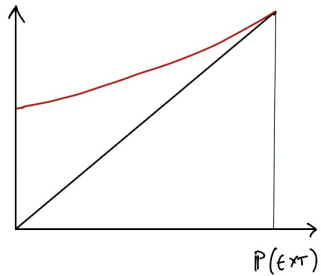


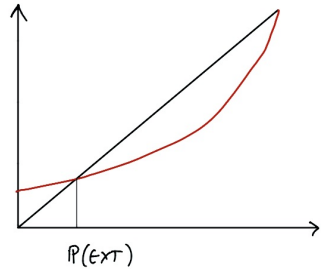


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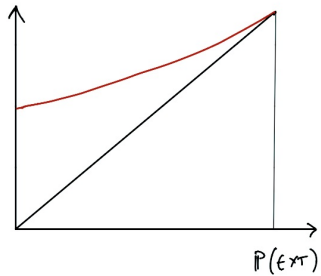
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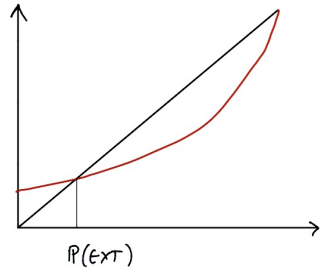
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$q_t \underset{t\to\infty}{\longrightarrow} 1$

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π_0 only fixed point in $[0, 1]$ of $\varphi(s) = s$

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Countable Markov chains

Two distinct situations

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(2) The chain does not have an equilibrium distribution
   (a) It is transient : each state is visited a finite number of times
   (b) It is recurrent : _____ infinitely many times but $\mathbb{E}\ \tau_x = \infty$ (average return time is infinite)

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Return times : $X_0 = x$

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Definition

State x is recurrent if $q(x) = \mathbb{P}(\tau_x < \infty) = 1$ and transient if $q(x) < 1$

Two kinds of recurrent states:

- Null recurrent if $\mathbb{E}\ \tau_x = \infty$
- Positive _____ $\mathbb{E}\ \tau_x < \infty$

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State x is recurrent $\quad \mathbb{P}(N_x = \infty) = 1$

transient $\quad \mathbb{P}(N_x = k) = q^k(x)(1 - q(x))$

Why?

Why? $\left(\tau_x^{(1)}, \tau_x^{(2)}, \ldots, \tau_x^{(k)}\right)$ distributed as $\left(W_x^{(1)}, W_x^{(1)} + W_x^{(2)}, \ldots, W_x^{(1)} + \ldots + W_x^{(k)}\right)$

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$$\mathbb{P}(N_x = k) = \mathbb{P}\left(W_x^{(1)} < \infty\ , \ldots, W_x^{(k)} < \infty\ , W_x^{(k+1)} = \infty\right) = q^k(x)[1 - q(x)]$$

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Note : If $\mathcal{X}$ is finite, then all states are necessarily recurrent

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Suppose one state x_0 is recurrent, then we show that all states are recurrent

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$\implies x$ recurrent

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- All positive _____

Example : SRW

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Why?

Recurrence and transience of SRW's

- 1D : $\mathbb{P}(X_{2n} = 0) = \binom{2n}{n} \left(\frac{1}{2}\right)^{2n} \sim \frac{1}{\sqrt{\pi n}}$

Recurrence and transience of SRW's

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- 2D : $\mathbb{P}(X_{2n} = 0) = \left[\binom{2n}{n} \left(\frac{1}{2}\right)^{2n}\right]^2 \sim \frac{1}{\pi n}$
- 3D+ : $\mathbb{P}(X_{2n} = 0) \sim \frac{3\sqrt{3}}{2(\pi n)^{3/2}}$

$$\mathbb{P}(\tau_0 < \infty) \approx 0.3405$$

Limit theorems

P irreducible : then there is at most one invariant distribution π

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Theorem (Ergodic theorem)

Irreducible chain with invariant dist. π

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} 1\{X_t = x\} = \pi(x) = \frac{1}{\mathbb{E}\tau_x}$$

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Shows that all states are positive recurrent

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If π exists, then $\pi(x) > 0 \quad \forall\, x \in \mathcal{X}$

Theorem (Ergodic theorem)

Irreducible chain with invariant dist. π

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} 1\{X_t = x\} = \pi(x) = \frac{1}{\mathbb{E}\tau_x}$$

Shows that all states are positive recurrent

Theorem (Convergence theorem)

Irreducible & aperiodic chain with invariant dist. π

$$\pi^{(t)} \longrightarrow \pi \quad \textit{i.e.} \quad \lim_{t\to\infty} \mathbb{P}(X_t = x) = \pi(x)$$

Stationary distribution criterion

Theorem

Irreducible (homogeneous) chain is positive recurrent iff there exists a stationary distribution

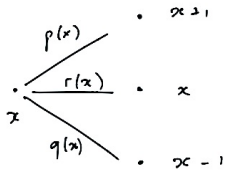
Moreover, inv. dist. π, when it exists, is unique & $\pi > 0$

Example : birth and death process

$\mathcal{X} = \{0, 1, 2, \ldots\}$

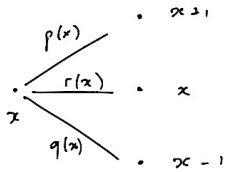
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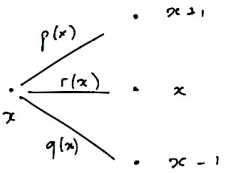
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Proposition

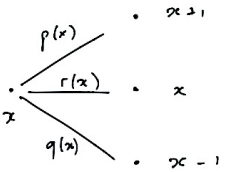
$\{X_t\}$ *positive recurrent (exists* π*) iff*

$$\sum_{x \geq 1} \lambda(x) < \infty \qquad \lambda(x) = \frac{p(0) \ldots p(x-1)}{q(1) \ldots q(x)}$$

In this case $\pi(0) = \left[1 + \sum_{x \geq 1} \lambda(x) \right]^{-1} \qquad \pi(x) = \lambda(x)\pi(0)$

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e.g. $p(x) = p$, $q(x) = q$, $r(x) = r \Longrightarrow$ recurrence iff $p < q$

Proof

Positive recurrence $\Leftrightarrow$ $\exists\ \pi$ s.t. $\pi P = \pi$ & $\pi(x) > 0 \quad \forall\ x \in \mathcal{X}$

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