Stats 318: Lecture # 16

Agenda: Bounds on mixing times of MCMC methods

- Mixing times for random coloring via Metrpolis Hastings
- Mixing times for Ising model via Gibbs sampling

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Metropolis chain

- Select site v uniformly at random
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$G = (V, E)$ with max degree Δ and $|V| = n$

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Improvement: $q > 2\Delta \Longrightarrow$ mixing time still on the order of $n \log n$

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$$\therefore \quad \mathbb{P}(\rho(X_1^x, X_1^y) = 0) = \frac{1}{n} \frac{q - |\mathcal{N}(v_0)|}{q} \geq \frac{q - \Delta}{q}$$

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$$\therefore \quad \mathbb{E}\left[\rho(X_1^x, X_1^y) - 1\right] \leq \frac{2\Delta}{nq} - \frac{q - \Delta}{nq} = \frac{3\Delta - q}{nq} = 1 - \frac{c_1}{n} < 1$$

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$$\therefore \quad t \geq \frac{1}{c_1} n [\log n + \log 1/\epsilon] \quad \Longrightarrow \quad d(t) \leq \epsilon$$

Reminder

$$d(t) = \max_x \|P^t(x,\cdot) - \pi\|_{\mathsf{TV}}$$
$$\overline{d}(t) = \max_{x,y} \|P^t(x,\cdot) - P^t(y,\cdot)\|_{\mathsf{TV}}$$

$$d(t) \le \overline{d}(t) \le 2d(t)$$

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$$\pi(x;\beta) \;\propto\; \exp\left(\beta \sum_{v\sim v'} x_v x_{v'}\right) \qquad \text{(edges counted once)}$$

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$$X(v) = \begin{cases} +1 & \text{wp } \dfrac{e^{\beta S(v)}}{e^{\beta S(v)} + e^{-\beta S(v)}} = \dfrac{1+\tanh(\beta S(v))}{2} \\ -1 & \text{wp } \dfrac{e^{-\beta S(v)}}{e^{\beta S(v)} + e^{-\beta S(v)}} = \dfrac{1-\tanh(\beta S(v))}{2} \end{cases} \qquad S(v) = \sum_{v':v'\sim v} x(v')$$

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- $\beta = \beta_c \Longrightarrow$ _____ polynomial in $|V|$

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(ii) *If every node has even degree*

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if $c_e(\beta) = 1 - \frac{\Delta}{2} \tanh(2\beta) > 0$

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Hamming distance

$$X(v) = \begin{cases} +1 & \text{if } U \leq \dfrac{1+\tanh(\beta S)}{2} \\ -1 & \text{else} \end{cases} \qquad Y(v) = \begin{cases} 1 & \text{if } U \leq \dfrac{1+\tanh(\beta(S+2))}{2} \\ -1 & \text{else} \end{cases}$$

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$$\begin{aligned} \therefore \quad \mathbb{E}\,\rho(X_1^x, X_1^y) &\leq 1 - \frac{1}{n} + \frac{1}{n} \sum_{v \in \mathcal{N}(v_0)} \frac{1}{2}\left[\tanh(\beta(S(v)+2)) - \tanh(\beta S(v))\right] \\ &\leq 1 - \frac{1}{n} + \frac{1}{n} \sum_{v \in \mathcal{N}(v_0)} \tanh(\beta) \leq 1 - \frac{1}{n} + \frac{\Delta}{n}\tanh(\beta) \end{aligned}$$

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$$\therefore \quad \mathbb{E}\,\rho(X_1^x, X_1^y) \leq \left[1 - \frac{1 - \Delta\tanh(\beta)}{n}\right] \leq e^{-c(\beta)/n} \quad \text{if } \Delta\tanh(\beta) < 1$$

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$$\therefore \qquad t \geq \frac{1}{c(\beta)} n \left[\log n + \log \frac{1}{\epsilon} \right] \qquad \Longrightarrow \quad d(t) \leq \epsilon$$

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Rest as before