Stats 318: Lecture # 10

Agenda:

- Gibbs sampling
- Examples

Gibbs sampling

Special case of Metropolis-Hastings algorithm over product space

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- Each $x \in \mathcal{X}$ has n components: $x = (x_1, \ldots, x_n)$
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(2) Sample X'_j from $\pi(X_j \mid X_{-j})$

(3) Update $(X_j, X_{-j}) \longleftarrow (X'_j, X_{-j})$

Formulation as a Metropolis iteration

- Proposal distribution

$$Q(x,y) = \begin{cases} \frac{1}{n}\pi(y_i \mid y_{-i}) & y_i \neq x_i,\ y_{-i} = x_{-i} \\ 0 & \text{otherwise and } y \neq x \\ 1 - \sum_{y:y\neq x} Q(y,x) & y = x \end{cases}$$

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- Consequence: chain converges to equilibrium

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- π unique equilibrium for both chains

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$$\therefore \quad \mathbb{P}(X_v = 1 \mid X_{-v} = x_{-v}) = \frac{1}{1 + e^{-2\beta g_v}}$$

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  - If $U < (1 + e^{-2\beta g_v})^{-1}$, $\quad X_v \longleftarrow +1$
    else $X_v \longleftarrow -1$