Math 2a Homework 6 Solutions

Problem 1. (*Handout, #2*) Let X be the number of heads observed. Then X is a binomial random variable with $n = 10$ trials and probability of success p. We wish to test the null hypothesis $H_0 : p = 0.5$ versus the alternative $H_a : p \neq 0.5$. The probability that we reject H_0 is $P(X = 0 \text{ or } X = 10) = (1-p)^{10} + p^{10}$. Therefore, by definition, the significance level of this test is $P(\text{ reject } H_0 \mid p = 0.5) = 0.5^{10} + 0.5^{10} \approx 0.002$ and if in fact $p = 0.1$ the power of the test is $P(\text{ reject } H_0 \mid p = 0.1) = 0.9^{10} + 0.1^{10} \approx 0.349$.

Problem 2. (*Handout, #3*) We need to approximate the distribution of the statistic $T = \dfrac{\overline{X} - \mu}{se(\overline{X})}\cdot$ If the size of the sample n is sufficiently large we can assume that T has standard normal distribution (note, however, that for smaller sample sizes the $t-$distribution with $n-1$ degrees of freedom can give a better approximation, especially if the data is close to normal). Since $\mu < \overline{X} + c^* se(\overline{X}) \iff -c^* se(\overline{X}) < \overline{X} - \mu \iff -c^* < T$, we can determine c^* from the condition $0.9 = P(T > -c^*) = 1 - \Phi(-c^*) = \Phi(c^*)$ which yields $c^* \approx 1.282$. Similarly, $\mu > \overline{X} - c^* se(\overline{X}) \iff c^* > T$ and from the condition $0.95 = P(T < c^*) = \Phi(c^*)$ we determine $c^* \approx 1.645$. For example, if $n = 500$ and we used the $t-$distribution with 499 degrees of freedom, we would get $c^* \approx 1.283$ and $c^* \approx 1.648$ for confidence levels 0.90 and 0.95 respectively.

Problem 3. (*Moore & McCabe, 7.40*) (a) This is an example of a matched pairs experiment design. We need to make sure each subject tries the two knobs in random order, for example by throwing a coin or using the random digits table at the end of the book.

(b) Let $\mu = \mu_{\text{left}} - \mu_{\text{right}}$ be the difference between the mean times required to complete the task with the left-threaded knob and with the right-threaded knob. Choose $H_0 : \mu = 0$ (there is no difference) and $H_a : \mu > 0$ (right-threaded knob is better, i.e. takes shorter time).

(c) Our sample consists of $n = 25$ differences between the left-thread and right-thread measurements and has $\overline{X} = 13.32$ and $S = \dfrac{1}{n-1}\displaystyle\sum_{i=1}^{n}(X_i - \overline{X})^2 \approx 22.936$. We compute the $t-$statistic $T = \dfrac{\overline{X} - 0}{S/\sqrt{n}} \approx 2.904$. Since T has (approximately) $t-$distribution with $n - 1 = 24$ degrees of freedom, we determine that the $p-$value of the test is $P(T \geq 2.904) \approx 0.004$. Thus, we can conclude that the data does not support H_0 at all significance levels ≥ 0.004 and it is very likely that the right-threaded knob is easier to operate.

Problem 4. (*Moore & McCabe, 7.64*) (a) The software reports the $p-$value for the two-sided alternative $\mu_1 \neq \mu_2$, which is equal to $2P(T \geq |t|) = 2P(T \geq t)$ when $t > 0$. Then the $p-$value for the one-sided alternative $\mu_1 > \mu_2$ is $P(T \geq 2.07) = 0.06/2 = 0.03 < 0.05$ so we would reject the null hypothesis $\mu_1 = \mu_2$ with $\alpha = 0.05$.

(b) No, because the $p-$value for the alternative $\mu_1 > \mu_2$ would be $P(T \geq -2.07) = 1 - 0.06 + 0.03 = 0.97$.

Problem 5. (*Moore & McCabe, 7.70*) (a) Let $H_0 : \mu_1 = \mu_2$ and $H_a : \mu_1 < \mu_2$, where μ_1 is the mean for the "positive test" group and μ_2 is the mean for the "other" group. We compute the two-sample $t-$statistic $T = \dfrac{\overline{X}_1 - \overline{X}_2}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}} \approx -7.337$. Next we approximate the distribution of T using the recipe for the degrees of freedom from p. 536 of the book which gives $df \approx 141$. The $p-$value $P(T \leq -7.337) \approx 7.8737 \times 10^{-12}$ is extremely small.

(b) The standard error is $\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}} \approx 52.471$ and $P(-t^* < T < t^*) = 0.95$ yields $t^* \approx 1.977$. Then the confidence interval for $\mu_1 - \mu_2$ is $(-385 - 52.471t^*, -385 + 52.471t^*) \approx (-488.732, -281.268)$.

(c) The "other" group consists not only of women who tested negative, but also of women who were not tested and might in fact be using drugs – if so, they could be reluctant to agree to a drug screening test. There are additional factors that need consideration, for example drug users might be likely to receive little or no prenatal care or might not have the financial means to maintain a healthy lifestyle. We should not conclude from this study that maternal drug use is a direct cause of low infant birth weights.

Problem 6. (*Handout, #7*) (a) Let Y be the event that a respondent answers "Yes" and let A_i be the event that he is responding to statement i. Then

$$r = P(Y) = P(Y \mid A_1)P(A_1) + P(Y \mid A_2)P(A_2) = qp + (1-q)(1-p) = (2p-1)q + 1 - p.$$

(b) From (a), we have $q = \frac{1}{2p-1}r + \frac{p-1}{2p-1}\cdot$ Note that we can assume $2p - 1 \neq 0$, because $p = \frac{1}{2}$ implies that $r = \frac{1}{2}$ and so r does not depend on q, hence the described protocol will provide no information about q and the experiment will be useless.

(c) Suppose the sample is $X_1, \ldots, X_n$, where $X_i = 1$ if the subject responds with "Yes" and $X_i = 0$ otherwise, so that $E(X_i) = r$. Then $R = \overline{X}$ and $E(R) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = r$. Let $Q = \frac{1}{2p-1}R + \frac{p-1}{2p-1}$, then $E(Q) = \frac{1}{2p-1}E(R) + \frac{p-1}{2p-1} = \frac{1}{2p-1}r + \frac{p-1}{2p-1} = q$.

(d) Suppose the size of the population is very large, so that we can assume the sampling is done with replacement and $X_1, \ldots, X_n$ are independent. Observe that $E(X_i^2) = E(X_i) = r$, so we have

$$\text{Var}(R) = \frac{1}{n^2}\sum_{i=1}^{n} E(X_i - r)^2 = \frac{1}{n^2}\left(\sum_{i=1}^{n} E(X_i^2) - 2r\sum_{i=1}^{n} E(X_i) + nr^2\right) = \frac{1}{n^2}(nr - 2nr^2 + nr^2) = \frac{r(1-r)}{n}\cdot$$

(e) From parts (c), (d) and (a), $\text{Var}(Q) = \frac{1}{(2p-1)^2}\text{Var}(R) = \frac{r(1-r)}{n(2p-1)^2} = \frac{(p+q-2pq)(1-p-q+2pq)}{n(2p-1)^2}\cdot$