SOLUTIONS TO HOMEWORK 3

Problem 1.

The total number of outcomes is 6^{12}, while the number of outcomes in which every face appears twice is the number of permutations of the set $\{1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6\}$ which is $\frac{(12)!}{(2!)^6}$. Thus the probability that every face appears twice is $\frac{\left(\frac{(12)!}{(2)!^6}\right)}{6^{12}} = 0.003438$.

Alternatively, by the multinomial distribution, we know that the probability that each face appears twice is

$$P(N_1 = 2, ..., N_6 = 2) = \frac{(12)!}{(2!)^6}\left(\frac{1}{6}\right)^{12} = 0.003438$$

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Problem 2.

Let n denote the number of students to be offered admission so that the probability of more than 240 students attending is below 5%. It is given that the probability of acceptance is $p = 0.55$. By the normal approximation of the binomial distribution we know that

$$P(\text{ up to } b \text{ successes }) \approx \Phi\left(\frac{b+\frac{1}{2}-\mu}{\sigma}\right).$$

In our case, we want the probability that the number of students is less than or equal to 240 to be 0.95. By the normal distribution table we get that $\Phi(z) > 0.95$ implies $z \geq 1.65$. Thus, we want $\frac{240-\mu}{\sigma} = 1.65$. Note that here $\mu = (0.55)n$ and $\sigma = \sqrt{(0.55)(0.45)n}$. Solving this we get $n = 407.15$ and thus if the admission is offered to 407 students then the probability that more than 240 students will attend is less than 5%.

Similarly, for the probability in question to be less than 10%, we need $\Phi(z) > 0.90$ which gives $z \geq 1.29$. Again, solving for n gives $n = 413.54$, and therefore if the admission is offered to 413 students then the probability that more than 240 students will attend is less than 10%.

Problem 3.

(a) Let us number the letters by the set $\{1, 2, ..., n\}$. Let

$$P_k = \sum_{i_1,i_2,...,i_k} \Pr(\text{Letters numbered } i_1, i_2, ..., i_k \text{ are in the correct envelopes}).$$

Then, by inclusion-exclusion princpal $\Pr(\text{At least one letter is in the correct envelope}) = \sum_{k=1}^{n}(-1)^{k+1}P_k$. To calculate this, first note that the number of ways in which letters numbered $i_1, i_2, ..., i_k$ are in the correct envelopes is $(n-k)!$ since the remaining $(n-k)$ letters could go into the remaining $(n-k)$ envelopes in any possible permutation. Thus,

$$\Pr(\text{Letters numbered } i_1, i_2, ..., i_k \text{ are in the correct envelopes}) = \frac{(n-k)!}{n!}.$$

This gives $P_k = \binom{n}{k}\frac{(n-k)!}{n!} = \frac{1}{k!}$. Therefore,

$$\Pr(\text{At least one letter is in the correct envelope}) = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!}.$$

(b) For large values of n this probability equals $1 - \sum_{k=0}^{n} \frac{(-1)^k}{k!} \approx 1 - e^{-1}$.

Problem 4.

(a) We toss the fair coin twice. We disregard the result if both the tosses are tails and continue to toss twice until we get a different result. Then, note that we end up getting two heads with

probability $\frac{1}{3}$ since $\Pr(HH|\{HH, HT, TH\}) = \frac{1}{3}$. Thus, we can call the trial in which there are two heads as HEADS and the others as TAILS. This simulates a biased coin with probability of heads being $\frac{1}{3}$.
(b) Similar to the last case, we toss the coin twice. In this case, we disregard the result if both the tosses are heads or if both of them are tails. We call the result HEADS if the first toss is heads and TAILS if the first toss is tails. Now, $\Pr(HT|\{TH, HT\}) = \frac{p(1-p)}{p(1-p)+(1-p)p} = \frac{1}{2}$ and thus we simulate a fair coin.

Problem 5.

(a) Note that we have $P(X+Y=n) = \sum_{k=0}^{n} P(X=k, Y=n-k)$. Since X and Y are independent we get $P(X=k, Y=n-k) = P(X=k)P(Y=n-k)$ for all $0 \le k \le n$. Thus, $P(X+Y=n) = \sum_{k=0}^{n} P(X=k)P(Y=n-k)$.
(b) The required probability is $P(X+Y=8) = \sum_{k=2}^{6} P(X=k)(Y=8-k)$. Note that since X and Y are sums of two dice, their range starts from 2. Now, $P(X=k) = \sum_{j=1}^{k-1} P(X_0 = j)P(X_1 = k-j)$ where X_0 and X_1 are the random variables correspoding to the two dice which add up to X. (Obviously, X_0 and X_1 are independent). But, $P(X_0 = i) = P(X_1 = i) = \frac{1}{6}$ for all $1 \le i \le 6$. Thus, for $k \le 7$, $P(X=k) = \sum_{j=1}^{k-1} \frac{1}{6} \cdot \frac{1}{6} = \frac{k-1}{36}$. Similarly, $P(Y=l) = \frac{l-1}{36}$. Hence, $P(X+Y=8) = \sum_{k=2}^{6} \left(\frac{k-1}{36}\right)\left(\frac{7-k}{36}\right) = \frac{1}{1296}\sum_{k=2}^{6}(k-1)(7-k) = \frac{35}{1296} = 0.027$.

Problem 6.

The answer is NO.

Let X be a random variable which equals 0 if a coin toss is heads, 1 otherwise. Let Y be another random variable which again equals 0 if a coin toss, independent of the previous one, is heads, and 1 otherwise. Thus, $P(X=0) = P(X=1) = P(Y=0) = P(Y=1) = \frac{1}{2}$. Let, Z be the random variable which equals 0 if the sum of X and Y is odd, 1 otherwise. Thus, $P(Z=0) = P(Z=1) = \frac{1}{2}$. Now, $P(Z=0|X=0, Y=0) = 0 \neq \frac{1}{2} = P(Z=0)$ and therefore X, Y and Z are not independent. However, by the description of X and Y they are independent. Further, X and Z are independent since $P(Z=\delta|X=\delta') = P(Z=\delta) = \frac{1}{2}$ and $P(X=\delta'|Z=\delta) = P(X=\delta') = \frac{1}{2}$ for all $\delta, \delta' \in \{0,1\}$. Similarly, Y and Z are independent. Thus, X, Y and Z are pairwise independent random variables but they are not independent overall.
