Ma 2a, Problem Set 2 Solutions

Due: Monday, October 13, 2003

1. This problem is open to interpretation. In order to solve the problem, assumptions must be made.

   Interpretation 1. Assume that the agent is equally likely to observe each pair of children. Let A be the event that the first two children the agent sees are girls, and let B be the event that all three children are girls. We have $P(B) = (1/2)^3 = 1/8$, $P(A \mid B) = 1$, $P(\text{not } B) = 7/8$, $P(A \mid \text{not } B) = 3/7 \cdot 1/3 = 1/7$, since there is a 3/7 chance that there are 2 girls, and for each of these, there is a 1/3 chance that the agent sees two girls. So, by Bayes' Rule, we have

   $$P(B \mid A) = \frac{P(A \mid B)P(B)}{P(A \mid B)P(B) + P(A \mid \text{not } B)P(\text{not } B)} = \frac{1 \cdot 1/8}{1 \cdot 1/8 + 1/7 \cdot 7/8} = \frac{1}{2}.$$

   So the probability that the third child is a girl is 1/2 and the probability that the third child is a boy is $1 - 1/2 = 1/2$.

   Interpretation 2. Suppose that we do not make the assumption that the agent sees a uniformly chosen random pair of the three children. Instead we assume that if there are at least two girls, then the agent will see a pair of girls. Let X be the number of girls out of the three children. Then

   $$P(X \geq 2) = \frac{\#\{\text{GGG}, \text{GGB}, \text{GBG}, \text{BGG}\}}{2^3} = \frac{1}{2}, \text{ and}$$
   $$P(X = 3) = \frac{1}{8}.$$

   So we compute the conditional probability of the third child being a girl as

   $$P(X = 3 \mid X \geq 2) = \frac{P(X = 3)}{P(X \geq 2)} = \frac{1/8}{1/2} = \frac{1}{4}.$$

   So the probability of the third child being a boy is 3/4.

2. a) Let A_r be the event that at least r students tell the rumor before it comes back to a student who has already heard the rumor. Then since it is true that if r people hear the rumor, then $r-1$ people have already heard it, we know that when A_r is true, A_{r-1} must also be true. So $A_r \cap A_{r-1} = A_r$. Let $p_r = P(A_r)$ as in the problem. From the definition of conditional probability, we have

   $$P(A_r \mid A_{r-1}) = \frac{P(A_r A_{r-1})}{P(A_{r-1})} = \frac{P(A_r)}{P(A_{r-1})} = \frac{p_r}{p_{r-1}}.$$

   By applying telescoping fractions, we have

   $$p_r = \frac{p_r}{p_1} = \frac{p_r}{p_{r-1}} \cdot \frac{p_{r-1}}{p_{r-2}} \cdot \ldots \cdot \frac{p_2}{p_1}.$$

Now it remains to calculate the conditional probability $P(A_r \mid A_{r-1}) = \frac{p_r}{p_{r-1}}$, for $r \geq 2$. If the event A_{r-1} has occurred this means that r students have been told the rumor already, two of which are the person who most recently heard the rumor and the person who told it to him/her. This means that there are $n-r$ people who could hear the rumor next and would not have already heard it, and there are $n-2$ people who might hear the rumor next. So $\frac{p_r}{p_{r-1}} = P(A_r \mid A_{r-1}) = \frac{n-r}{n-2}$, and

$$p_r = \frac{n-r}{n-2} \cdot \frac{n-r+1}{n-2} \cdot \dots \cdot \frac{n-3}{n-2} = \frac{(n-r)(n-r+1)\dots(n-3)}{(n-2)^{r-2}}.$$

b) It is possible to compute this probability numerically by multiplying the fractions $\frac{p_r}{p_{r-1}}$ as above. It is also possible to approximate it using Stirling's formula, or other methods. The exact value is ≈ 0.2446.

3. The outcome can be considered as an ordered sequence of 5 student names.

a) In this case, the outcomes allow replacement. There are $30^5 = 24300000$ possible outcomes.

b) In this case, the outcomes do not allow replacement. There are $30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 = 17100720$ possible outcomes.

4. a) All of the offspring will be tall with purple flowers (because they have genotype (Ts, Pw)).

b) There are four ways for the height genes from the plant to combine. These ways are TT, Ts, sT, and ss. Each of these four are equally likely. But Ts, sT are equivalent so we will write both of them as Ts. We have $P(\text{TT}) = P(\text{ss}) = 1/4$ and $P(\text{Ts}) = 1/2$. Similarly, for the flower color genes we have $P(\text{PP}) = P(\text{ww}) = 1/4$ and $P(\text{Pw}) = 1/2$. So in all there are $3 \times 3 = 9$ possible genetic combinations, one for each pair of one genotype for height and one genotype for flower color. Because the genes are independent, we have the following genotype probabilities:

$$\begin{aligned} P(\text{TT},\text{PP}) = P(\text{TT},\text{ww}) = P(\text{ss},\text{PP}) = P(\text{ss},\text{ww}) &= 1/4 \cdot 1/4 = 1/16, \\ P(\text{Ts},\text{PP}) = P(\text{Ts},\text{ww}) = P(\text{TT},\text{Pw}) = P(\text{ss},\text{Pw}) &= 1/4 \cdot 1/2 = 1/8, \\ P(\text{Ts},\text{Pw}) &= 1/2 \cdot 1/2 = 1/4. \end{aligned}$$

We can also find the phenotype probabilities:

$$\begin{aligned} P(\text{Tall},\text{Purple}) &= P(\text{TT},\text{PP}) + P(\text{TT},\text{Pw}) + P(\text{Ts},\text{PP}) + P(\text{Ts},\text{Pw}) \\ &= 1/16 + 1/8 + 1/8 + 1/4 = 9/16, \\ P(\text{Tall},\text{White}) &= P(\text{TT},\text{ww}) + P(\text{Ts},\text{ww}) = 1/16 + 1/8 = 3/16, \\ P(\text{Short},\text{Purple}) &= P(\text{ss},\text{PP}) + P(\text{ss},\text{Pw}) = 1/16 + 1/8 = 3/16, \\ P(\text{Short},\text{White}) &= P(\text{ss},\text{ww}) = 1/16. \end{aligned}$$

c) The probability that a given offspring is tall with purple flowers is 9/16. So we have a binomial distribution with $n = 10$ and $p = 9/16$, and we need to find the probability of at least 2 successes. This can be done by finding the probability of 0 or 1 successes and subtracting that quantity from 1. By the binomial distribution, we have $P(0) = (1-p)^{10} = (7/16)^{10} \approx 0.0003$, and $P(1) = \binom{10}{1} p(1-p)^9 = 10(9/16)(7/16)^9 \approx 0.0033$. So the answer is

$$1 - P(0) - P(1) \approx 0.9964.$$

5. To find $P(X=1)$, just notice that 11 out of the 66 balls cause this event to be true and the rest do not, so $P(X=1)=11/66=1/6\approx 0.1667$.

   To do the other cases, we note that after a team has "won" the lottery, it may be assumed that all of the corresponding balls for that team are removed. Suppose there are n balls left and a team with k balls wins. Then in the subsequent rounds, if a ball is drawn for the same team, it is removed and then a new ball is drawn. So this means in the next round there are only $n-k$ balls left that can advance the lottery.

   To find $P(X=2)$, suppose that the team with $k<11$ balls in the urn wins the first drawing. The probability of that event is $k/66$. When the second ball is drawn, only $66-k$ viable balls remain, so the chance that the team with the worst record wins the second drawing is $\frac{k}{66}\cdot\frac{11}{66-k}$. Cancelling this product leaves $\frac{1}{6}\cdot\frac{k}{66-k}$. So in all, we have

$$P(X=2)=\frac{1}{6}\sum_{k=1}^{10}\frac{k}{66-k}\approx 0.1556.$$

   To find $P(X=3)$, suppose that the team with $k<11$ balls in the urn wins the first drawing and the team with $m<11$ and $m\neq k$ balls in the urn wins the second drawing. The probability of these events is $\frac{k}{66}\cdot\frac{m}{66-k}$. Given that these events occur, the probability that the team with the worst record wins the third drawing is $\frac{11}{66-k-m}$. Cancelling this product leaves $\frac{1}{6}\cdot\frac{km}{(66-k)(66-k-m)}$. So in all we have

$$P(X=3)=\frac{1}{6}\sum_{\substack{1\le m,k\le 10\\ m\neq k}}\frac{km}{(66-k)(66-k-m)}\approx 0.1435.$$

   Now take a deep breath. To find $P(X=4)$, notice that it is not possible for X to be greater than 4, so $P(X=k)=0$ when $k>4$. This also means that $P(X=1)+P(X=2)+P(X=3)+P(X=4)=1$, implying

$$P(X=4)=1-P(X=1)-P(X=2)-P(X=3)\approx 0.5342.$$