Homework 2

Due February 19

1. Suppose that f is bandlimited to $[-\pi/T, \pi/T]$.

   (a) Give the filter $\phi(t)$ such that for any f,

   $$\tilde{f}(nT) = \frac{1}{T}\int_{(n-1/2)T}^{(n+1/2)T} f(t)\,dt = f * \phi(nT)$$

   Solution: By definition

   $$f * \phi(nT) = \int_{\mathbb{R}} f(s)\phi(nT-s)ds.$$

   On the other hand, if we denote by $I_{[a,b]}$ the indicator function on the interval $[a,b]$, we can write

   $$\frac{1}{T}\int_{(n-1/2)T}^{(n+1/2)T} f(t)\,dt = \int_{\mathbb{R}} f(s)\frac{1}{T}I_{[(n-1/2)T,(n+1/2)T]}(s)ds = \int_{\mathbb{R}} f(s)\frac{1}{T}I_{[-T/2,T/2]}(nT-s)ds.$$

   Since the two previous equations must hold for any f, this implies that $\phi(t) = \frac{1}{T}I_{[-T/2,T/2]}(t)$, which is the boxcar function.

   (b) Show that $\tilde{f}(t) = f * \phi(t)$ can be recovered from $\{\tilde{f}(nT)\}_{n\in\mathbb{Z}}$ with an interpolation formula.

   Solution: For legibility, we just assume $T = 1$ since the extension to arbitrary T is straightforward. By the convolution theorem, we have

   $$\hat{\tilde{f}}(\omega) = \hat{f}(\omega)\hat{\phi}(\omega).$$

   Since $\hat{f}$ has support on $[-\pi,\pi]$, we have that $\hat{\tilde{f}}$ also has support in $[-\pi,\pi]$. Therefore, we can apply the Shannon sampling theorem to $\tilde{f}$ and obtain

   $$\tilde{f}(t) = \sum_{n\in\mathbb{Z}} \tilde{f}(n)h(t-n), \quad h(t) = \operatorname{sinc}(t).$$

   The formula for general T is of course

   $$\tilde{f}(t) = \sum_{n\in\mathbb{Z}} \tilde{f}(nT)h_T(t-nT), \quad h_T(t) = \operatorname{sinc}\left(\frac{t}{T}\right).$$

   (c) Reconstruct f from $\tilde{f}$ by inverting ϕ.

   Solution: Applying the Fourier transform and the convolution theorem to the equation in the statement of the previous part gives that $\hat{\tilde{f}} = \hat{f}\hat{\phi}$. Therefore, $\hat{f} = \hat{\tilde{f}}/\hat{\phi}$. Applying the inverse Fourier transform to both sides gives then that

   $$f(t) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\hat{\tilde{f}}(\omega)}{\hat{\phi}(\omega)}e^{it\omega}d\omega.$$

To simplify this expression, first note that $\hat{\phi}(\omega) = \text{sinc}(Tw/(2\pi))$ since it is a scaled boxcar function. Second, note that we have from the result of the previous part that

$$\hat{\tilde{f}}(\omega) = \sum_{n \in \mathbb{Z}} \tilde{f}(nT) e^{-i\omega nT} T I(|T\omega| < \pi)$$

since $\widehat{h_T(\cdot - nT)}(\omega) = e^{-i\omega nT} T I(|T\omega| < \pi)$. Plugging in what we have,

$$f(t) = \sum_{n \in \mathbb{Z}} \tilde{f}(nT) \frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} \frac{e^{i(t-nT)\omega}}{\text{sinc}(T\omega/(2\pi))} T d\omega$$

which is reconstructing f using only $\{\tilde{f}(nT)\}_{n \in \mathbb{Z}}$.
If we let again $T = 1$ for legibility and we consider g whose Fourier transform is $I(|\omega| \leq \pi)/\text{sinc}(\omega/(2\pi))$, this reconstruction formula can be written more transparently as

$$f(t) = \sum_{n \in \mathbb{Z}} \tilde{f}(n) g(t - n)$$

using the Fourier inversion formula.

(d) Suppose I observe

$$\tilde{y}(nT) = \tilde{f}(nT) + \epsilon(nT),$$

where $\epsilon(nT)$ is a small error term (after all, it is impossible to record values with infinite precision) in the sense that $\sum_{n \in \mathbb{Z}} \epsilon^2(nT) \leq \delta^2$ for perhaps a small value of δ. I use the reconstruction formula above to reconstruct a signal $y(t)$. Can $y(\cdot)$ be very different from $f(\cdot)$? In particular, can the integrated MSE

$$\int_{\mathbb{R}} |f(t) - y(t)|^2 \, dt$$

be large? Explain why or why not.

Solution: No, it can't. To show this, first note that the approximation error is

$$f(t) - y(t) = \sum_{n \in \mathbb{Z}} \epsilon(n) g(t - n)$$

assuming again $T = 1$ for legibility. We can then bound the MSE as

$$\begin{aligned}
\int_{\mathbb{R}} |f(t) - y(t)|^2 \, dt &= \int_{\mathbb{R}} \left| \sum_{n \in \mathbb{Z}} \epsilon(n) g(t - n) \right|^2 dt \\
&\leq \int_{\mathbb{R}} \sum_{n \in \mathbb{Z}} \epsilon(n)^2 \, |g(t - n)|^2 \, dt \\
&= \sum_{n \in \mathbb{Z}} \epsilon(n)^2 \int_{\mathbb{R}} |g(t - n)|^2 \, dt \\
&= \sum_{n \in \mathbb{Z}} \epsilon(n)^2 \int_{\mathbb{R}} |g(t)|^2 \, dt \\
&\leq \delta^2 \int_{\mathbb{R}} |g(t)|^2 \, dt.
\end{aligned}$$

Using the Parseval-Plancherel theorem, we finally see that

$$\int_{\mathbb{R}} |g(t)|^2 \, dt = \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{g}(\omega)|^2 \, d\omega = \frac{1}{2\pi} \int_{|\omega| \leq \pi} \frac{d\omega}{\text{sinc}(\omega/(2\pi))^2}$$

which is bounded since $\mathrm{sinc}(\omega/(2\pi))^2$ is continuous and non-zero in the integration interval. The proof for general T is analogous.

2. We want to compute numerically the Fourier transform of $f(t)$. Let f_d be the discrete-time signal $f_d[n] = f(nT)$ (we use square brackets to indicate that the argument is discrete) and f_p be the periodization $f_p[n] = \sum_{k=-\infty}^{\infty} f_d[n-kN]$.

   (a) Show that the FDT of f_p is related to the Fourier series of $f_d[n]$ and the Fourier transfrom of $f(t)$ by

$$\hat{f}_p[k] = \hat{f}_d\Big(\frac{2\pi k}{N}\Big) = \frac{1}{T}\sum_{\ell=-\infty}^{\infty} \hat{f}\Big(\frac{2\pi k}{NT} - \frac{2\pi \ell}{T}\Big).$$

Solution: Recall that

$$f_d(t) = \sum_{n\in\mathbb{Z}} f(nT)\delta(t-nT) \quad \Longrightarrow \quad \hat{f}_d(\omega) = \sum_{n\in\mathbb{Z}} f(nT)e^{-i\omega nT}$$

By a direct computation

$$\begin{aligned}
\hat{f}_p[k] &= \sum_{n=0}^{N-1} f_p[n]e^{-\frac{i2\pi kn}{N}} \\
&= \sum_{m=-\infty}^{\infty}\sum_{n=0}^{N-1} f_d[n-mN]e^{-\frac{i2\pi kn}{N}} \\
&= \sum_{m=-\infty}^{\infty}\sum_{n=-mN}^{N-1-mN} f_d[n]e^{-\frac{i2\pi k(n+mN)}{N}} \\
&= \sum_{m=-\infty}^{\infty}\sum_{n=-mN}^{N-1-mN} f(nT)e^{-\frac{i2\pi kn}{N}} \\
&= \sum_{n=-\infty}^{\infty} f(nT)e^{-\frac{i2\pi k}{NT}nT} \\
&= \hat{f}_d\left(\frac{2\pi k}{NT}\right),
\end{aligned}$$

and, by the aliasing formula

$$\hat{f}_p[k] = \hat{f}_d\left(\frac{2\pi k}{NT}\right) = \frac{1}{T}\sum_{n\in\mathbb{Z}} \hat{f}\left(\frac{2\pi k}{NT} - \frac{2\pi}{T}n\right)$$

   (b) Suppose that $|f(t)|$ and $|\hat{f}(\omega)|$ are negligible when $t \notin [-t_0, t_0]$ and $\omega \notin [-\omega_0, \omega_0]$. Relate N and T to t_0 and ω_0 so that one can compute an approximate value of $\hat{f}(\omega)$ for all $\omega \in \mathbb{R}$ by interpolating the samples $\hat{f}_p[k]$. Is it possible to compute exactly $\hat{f}(\omega)$ with such an interpolation formula.

Solution: If f is approximately time-limited on $[-t_0, t_0]$ we can use the approximation

$$f(t) \approx \frac{1}{\sqrt{2t_0}}\mathbb{I}\{|t| \le t_0\}\sum_{k\in\mathbb{Z}} c_k e^{\frac{i\pi}{t_0}kt},$$

with

$$c_k = \frac{1}{\sqrt{2t_0}}\int_{-t_0}^{t_0} f(t)e^{-\frac{i\pi}{t_0}kt}\,dt \approx \frac{1}{\sqrt{2t_0}}\int f(t)e^{-\frac{i\pi}{t_0}kt}\,dt = \frac{1}{\sqrt{2t_0}}\hat{f}\left(\frac{\pi}{t_0}k\right).$$

Since

$$\int \mathbb{I}\{|t| \le t_0\} e^{-i\omega t}\, dt = 2t_0 \operatorname{sinc}\left(\frac{t_0}{\pi}\omega\right),$$

taking Fourier transform and using the above we obtain

$$\hat{f}(\omega) \approx \frac{1}{2t_0} \sum_{k\in\mathbb{Z}} \hat{f}\left(\frac{\pi}{t_0}k\right)\left[2t_0 \operatorname{sinc}\left(\frac{t_0}{\pi}\left(\omega - \frac{\pi}{t_0}k\right)\right)\right] = \sum_{k\in\mathbb{Z}} \hat{f}\left(\frac{\pi}{t_0}k\right) \operatorname{sinc}\left(\frac{t_0}{\pi}\omega - k\right).$$

Since f is approximately bandlimited on $[-\omega_0, \omega_0]$ we can further approximate $\hat{f}$ as

$$\hat{f}(\omega) \approx \sum_{|k|\le \frac{\omega_0 t_0}{\pi}} \hat{f}\left(\frac{\pi}{t_0}k\right) \operatorname{sinc}\left(\frac{t_0}{\pi}\omega - k\right).$$

This shows one wants to sample $\hat{f}$ at points of the form $\pi k/t_0$ for $\pi|k| \le \omega_0 t_0$ on the frequency domain and use sinc interpolation to obtain approximations of $\hat{f}$ for any $\omega \in \mathbb{R}$. In order to do this, we use the result in part (a). If $NT = 2t_0$ then

$$\hat{f}_p[k] = \frac{1}{T}\sum_{n\in\mathbb{Z}} \hat{f}\left(\frac{\pi}{t_0}k - \frac{\pi N}{t_0}n\right) = \frac{1}{T}\hat{f}\left(\frac{\pi}{t_0}k\right) + \frac{1}{T}\sum_{n\in\mathbb{Z}} \hat{f}\left(\frac{\pi}{t_0}(k - nN)\right),$$

and

$$f_p[n] = \sum_{m\in\mathbb{Z}}^{\infty} f_d[n - mN] = f(nT) + \sum_{m\in\mathbb{Z}\setminus\{0\}} f((n-mN)T) = f\left(\frac{2t_0}{N}n\right) + \sum_{m\in\mathbb{Z}\setminus\{0\}} f\left(2t_0\left(\frac{n}{N} - m\right)\right).$$

Using the fact that f is approximately time-limited on $[-t_0, t_0]$, if $|n| \le N/2$ then

$$f_p[n] \approx f(nT),$$

and using the fact that $\hat{f}$ is approximately bandlimited on $[-\omega_0, \omega_0]$, if $N > 2\omega_0 t_0/\pi$ we see that

$$\hat{f}_p[k] \approx \frac{1}{T}\hat{f}\left(\frac{\pi}{t_0}k\right).$$

In summary

i. Choose N (say, even) such that $2\omega_0 t_0 < \pi N$ and T such that $NT = 2t_0$.

ii. Sample f at the points $\{nT\}$ for $-N/2 \le n \le N/2 - 1$.

iii. Compute the DFT $\hat{f}_d$ of the discrete signal of length N

$$f_d[n] = f(nT), \qquad -N/2 \le n \le N/2 - 1.$$

iv. Use sinc interpolation

$$\hat{f}(\omega) \approx T \sum_{k=-N/2}^{N/2} \hat{f}_p[k] \operatorname{sinc}\left(\frac{t_0}{\pi}\omega - k\right).$$

It is clear we cannot compute $\hat{f}$ exactly with this formula. The reason lies on the result of part (a). We can write

$$\begin{aligned}
\sum_{n=0}^{N-1} f(nT)e^{-\frac{i2\pi}{N}kn} &= \hat{f}_p[k] - \sum_{n=0}^{N-1}\left(\sum_{m\in\mathbb{Z}\setminus\{0\}} f((n-m)T)\right) e^{-\frac{i2\pi}{N}kn} \\
&= \frac{1}{T}\sum_{n\in\mathbb{Z}} \hat{f}\left(\frac{2\pi k}{NT} - \frac{2\pi}{T}n\right) - \sum_{n=0}^{N-1}\left(\sum_{m\in\mathbb{Z}\setminus\{0\}} f((n-m)T)\right) e^{-\frac{i2\pi}{N}kn},
\end{aligned}$$

and consequently computing the DFT of *any* finite sequence of samples will cause errors because of aliasing effects *both in time and frequency.* Since a function[1] cannot be both simultaneously time-limited and bandlimited, there is no way to eliminate these approximation errors.

(c) Let

$$f(t) = \left(\frac{\sin(\pi t)}{\pi t}\right)^4.$$

What is the support of $\hat{f}$? Sample f appropriately and compute $\hat{f}$ numerically with an FFT algorithm.

Solution: By the convolution theorem

$$\hat{f}(\omega) = \frac{1}{(2\pi)^3}\,\mathbb{I}\{|t| \le \pi\} * \mathbb{I}\{|t| \le \pi\} * \mathbb{I}\{|t| \le \pi\} * \mathbb{I}\{|t| \le \pi\},$$

and therefore $\mathbf{supp}(\hat{f}) \subset [-4\pi, 4\pi]$. Consequently f is bandlimited. On the other hand, note that

$$|f(t)| \le \frac{1}{(\pi t)^4},$$

for, say, $t > 1$. From this it follows that we can choose t_0 depending on how much error are we willing to tolerate in our approximations. The above implies that $|f(t)| < 10^{-d}$ for $d > 0$ outside $[-t_0, t_0]$ if $t_0 > e^{-d/4}/\pi$. For $d = 6$ we can pick, for instance, $t_0 = 3/2$. From this we have

$$2\frac{\omega_0 t_0}{\pi} = 2\frac{(4\pi)(3/2)}{\pi} = 12 < N,$$

so that with $N = 14$ we can choose

$$T = \frac{2t_0}{N} = \frac{3}{14}.$$

The resulting approximation can be seen in Fig. 1.

3. Suppose I want to compute

$$\hat{f}(\omega) = \sum_{t=0}^{N-1} f(t)e^{-i2\pi\omega t}$$

on a grid of the form $\omega = k/2N$, where $k = 0, \ldots, 2N-1$. Explain how I would use the FFT to do this. Please describe an algorithm which is as efficient as possible.

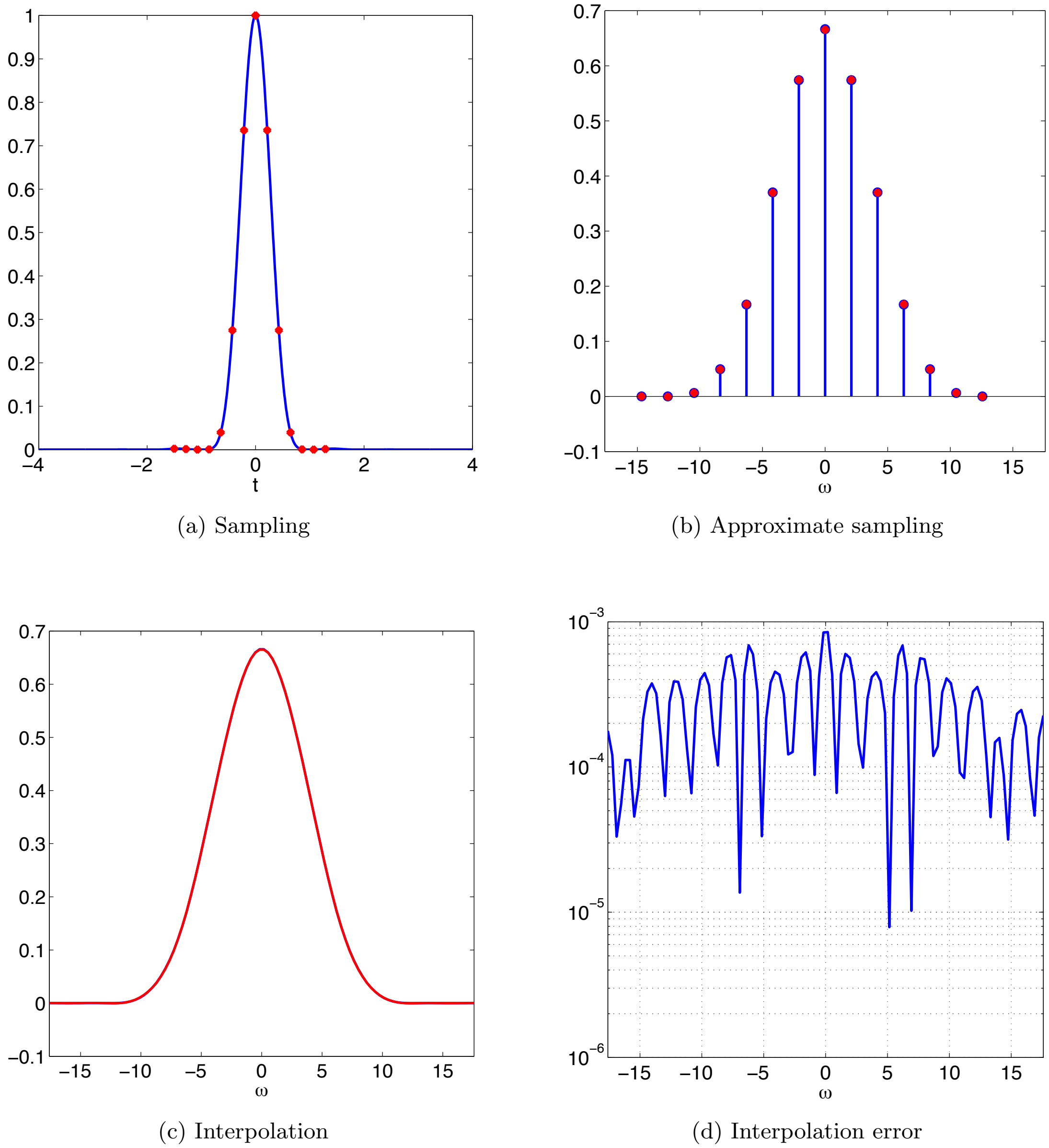
Suppose I want to compute the derivative $d\hat{f}(\omega)/d\omega$ on the same grid. How would I do this? How would I compute the second derivative?

Solution: One possible solution is to zero pad f so that $f(t) = 0$ for $t = N, \ldots, 2N-1$. Then we have that

$$\begin{aligned}\hat{f}\left(\frac{k}{2N}\right) &= \sum_{t=0}^{N-1} f(t)e^{-i2\pi\frac{kt}{2N}} \\ &= \sum_{t=0}^{2N-1} f(t)e^{-i2\pi\frac{kt}{2N}}\end{aligned}$$

and the last equation can be computed efficiently by using the FFT with complexity $O(2N\log(2N)) = O(2N\log N + 2N\log 2) = O(N\log N)$.

[1]In fact, a function that is bandlimited cannot vanish on an interval.



(a) Sampling (b) Approximate sampling

(c) Interpolation (d) Interpolation error

Figure 1: (a) The function f under consideration (in blue) and sampling points (in red); (b) Approximation of the Fourier coefficients using the DFT of the finite discrete signal (in red) and true samples (in blue); (c) Interpolation of the true Fourier transform $\hat{f}$ (in blue) using sinc interpolation (in red). Both curves overlap; (d) Error between the interpolation and true Fourier transform (in logarithmic scale).

Alternatively, is also possible to compute the odd indices and even indices separately. For even indices,

$$\begin{aligned}\hat{f}\left(\frac{2k}{2N}\right) &= \sum_{t=0}^{N-1} f(t)e^{-i2\pi\frac{2kt}{2N}} \\ &= \sum_{t=0}^{N-1} f(t)e^{-i2\pi\frac{kt}{N}}\end{aligned}$$

which can be obtained from FFT of $\{f(t)\}_{t=0}^{N-1}$ in $O(N \log N)$. For odd indices,

$$\begin{aligned}\hat{f}\left(\frac{2k+1}{2N}\right) &= \sum_{t=0}^{N-1} f(t)e^{-i2\pi\frac{(2k+1)t}{2N}} \\ &= \sum_{t=0}^{N-1} f(t)e^{-i2\pi\frac{t}{2N}}e^{-i2\pi\frac{kt}{N}},\end{aligned}$$

so we can apply the FFT on the signal $\{f(t)e^{-i2\pi\frac{t}{2N}}\}_{t=0}^{N-1}$ which will require $O(N \log N + N)$ computations including computing the new signal. Hence, the total complexity of the second solution is also $O(2N \log N + N) = O(N \log N)$.

To compute the derivative $d\hat{f}(\omega)/d\omega$, just note that $d\hat{f}(\omega)/d\omega = -2\pi i\hat{g}(w)$ where $g(t) = tf(t)$, so we can apply the previously described process to g instead of f and then multiply the answer by $-2\pi i$. Computing the second derivative is analogous, but applying the process to $h(t) = t^2 f(t)$ and multiplying the answer by the factor $(-2\pi i)^2$. The complexity in both cases is again $O(N \log N)$.