Homework 2

Due February 19

1. Suppose that $\hat{f}$ is bandlimited to $[-\pi/T, \pi/T]$.

   (a) Give the filter $\phi(t)$ such that for any f,

   $$\tilde{f}(nT) = \frac{1}{T}\int_{(n-1/2)T}^{(n+1/2)T} f(t)\,dt = f * \phi(nT)$$

   (b) Show that $\tilde{f}(t) = f * \phi(t)$ can be recovered from $\{\tilde{f}(nT)\}_{n\in\mathbb{Z}}$ with an interpolation formula.

   (c) Reconstruct f from $\tilde{f}$ by inverting ϕ.

   (d) Suppose I observe

   $$\tilde{y}(nT) = \tilde{f}(nT) + \epsilon(nT),$$

   where $\epsilon(nT)$ is a small error term (after all, it is impossible to record values with infinite precision) in the sense that $\sum_{n\in\mathbb{Z}} \epsilon^2(nT) \le \delta^2$ for perhaps a small value of δ. I use the reconstruction formula above to reconstruct a signal $y(t)$. Can $y(\cdot)$ be very different from $f(\cdot)$? In particular, can the integrated MSE

   $$\int_{\mathbb{R}} |f(t) - y(t)|^2\,dt$$

   be large? Explain why or why not.

2. We want to compute numerically the Fourier transform of $f(t)$. Let f_d be the discrete-time signal $f_d[n] = f(nT)$ (we use square brackets to indicate that the argument is discrete) and f_p be the periodization $f_p[n] = \sum_{k=-\infty}^{\infty} f_d[n-kN]$ (so we can think of f_p as a signal of length N).

   (a) Show that the DFT of f_p is related to the Fourier series of $f_d[n]$ and the Fourier transform of $f(t)$ by

   $$\hat{f}_p[k] = \hat{f}_d\Big(\frac{2\pi k}{N}\Big) = \frac{1}{T}\sum_{\ell=-\infty}^{\infty} \hat{f}\Big(\frac{2\pi k}{NT} - \frac{2\pi\ell}{T}\Big).$$

   (b) Suppose that $|f(t)|$ and $|\hat{f}(\omega)|$ are negligible when $t \notin [-t_0, t_0]$ and $\omega \notin [-\omega_0, \omega_0]$. Relate N and T to t_0 and ω_0 so that one can compute an approximate value of $\hat{f}(\omega)$ for all $\omega \in \mathbb{R}$ by interpolating the samples $\hat{f}_p[k]$. Is it possible to compute exactly $\hat{f}(\omega)$ with such an interpolation formula?

   (c) Let

   $$f(t) = \Big(\frac{\sin(\pi t)}{\pi t}\Big)^4.$$

   What is the support of $\hat{f}$? Sample f appropriately and compute $\hat{f}$ numerically with an FFT algorithm.

3. Suppose I want to compute

   $$\hat{f}(\omega) = \sum_{t=0}^{N-1} f(t)e^{-i2\pi\omega t}$$

   on a grid of the form $\omega = k/2N$, where $k = 0, \ldots, 2N-1$. Explain how I would use the FFT to do this. Please describe an algorithm which is as efficient as possible.

   Suppose I want to compute the derivative $d\hat{f}(\omega)/d\omega$ on the same grid. How would I do this? How would I compute the second derivative?