Homework 1 Solutions

1. (a) First,

$$c_0 = \int_0^1 f(t)\ dt = 0.$$

For $k \neq 0$, note that $t - 1/2 = f(t + 1/2)$ for $t \in [-1/2, 1/2)$, and so

$$\int_0^1 f(t)e^{-i2\pi kt}\, dt = \int_{-1/2}^{1/2} f(t+1/2)e^{-i2\pi k(t+1/2)}\, dt = (-1)^k \int_{-1/2}^{1/2} \left(t - \frac{1}{2}\right) e^{-i2\pi kt}\, dt.$$

Since

$$\begin{aligned}
\int_{-1/2}^{1/2} \left(t - \frac{1}{2}\right) e^{-i2\pi kt}\, dt &= \int_{-1/2}^{1/2} te^{-i2\pi kt}\, dt \\
&= -\frac{1}{i2\pi k} te^{-i2\pi kt}\Big|_{-1/2}^{1/2} + \frac{1}{i2\pi k}\int_{-1/2}^{1/2} e^{-i2\pi kt}\, dt \\
&= -\frac{1}{i2\pi k}\cos(\pi k) \\
&= \frac{(-1)^k i}{2\pi k}.
\end{aligned}$$

(b) By the Parseval-Plancherel theorem,

$$\|f - S_n(f)\|_{L^2}^2 = \|\hat{f} - \hat{S_n}(f)\|_{L^2}^2 = \frac{1}{2\pi^2}\sum_{k>n}\frac{1}{k^2} \leq \frac{1}{2\pi^2}\int_{n+1}^{\infty}\frac{dk}{k^2}.$$

And we can approximate the sum by an integral:

$$\frac{1}{n+1} = \int_{n+1}^{\infty}\frac{dx}{x^2} \leq \sum_{k>n}\frac{1}{k^2} \leq \int_n^{\infty}\frac{dx}{x^2} = \frac{1}{n}.$$

In summary,

$$\frac{1}{\sqrt{2\pi^2}}\frac{1}{\sqrt{n+1}} \leq \|f - S_n(f)\|_{L^2} \leq \frac{1}{\sqrt{2\pi^2}}\frac{1}{\sqrt{n}}.$$

(c) In this case,

$$c_k = -\int_0^{1/2} e^{-i2\pi kt}\, dt + \int_{1/2}^1 e^{-i2\pi kt}\, dt = -(1 - e^{-i\pi k})\int_0^{1/2} e^{-i2\pi kt}\, dt = -(1-(-1)^k)^2\frac{1}{i2\pi k},$$

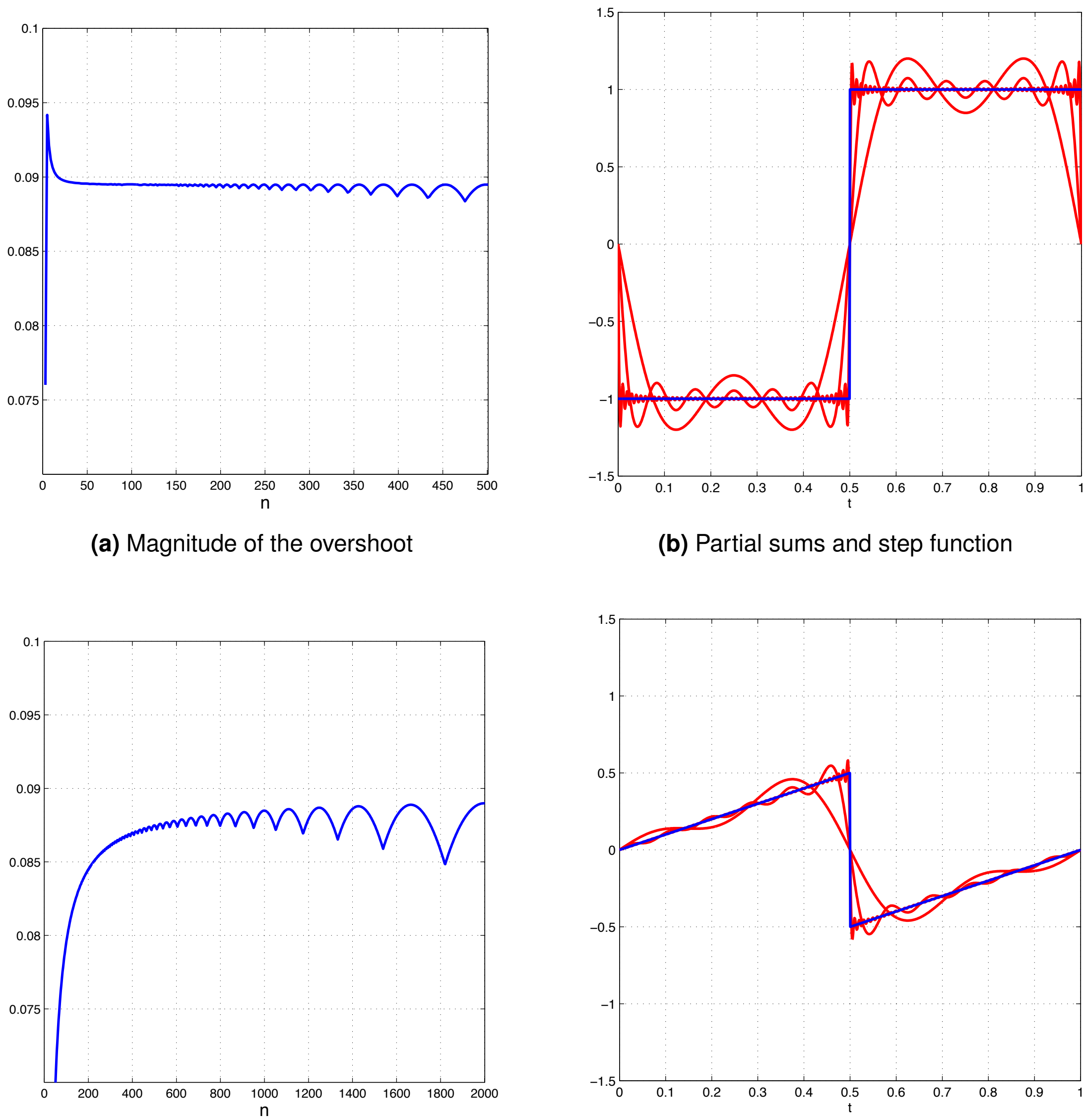
and

$$c_k = \begin{cases} 0 & \text{if } k \text{ is even,} \\ \frac{2i}{\pi k} & \text{if } k \text{ is odd.} \end{cases}$$

See Fig. 1b to see a sample of partial sums. See Fig. 1a for the corresponding Gibbs error

$$\frac{\text{max of } S_n(f) \text{ near jump} - \text{min of } S_n(f) \text{ near jump}}{\text{actual size of jump}},$$

which stabilizes at just under 9% as n increases.



(a) Magnitude of the overshoot

(b) Partial sums and step function

(c) Magnitude of the overshoot

(d) Partial sums and the sawtooth signal

Figure 1: Figures for Problem 1(d) and (e). The oscillations in Figures (a) and (c) are discretization artifacts.

(d) Now use the coefficients from (a). See Fig. 1c and Fig. 1d. Once again, the overshoot remains approximately 9% as n grows large.

(e) *Bonus question.* Since even coefficients vanish, $S_n(f) = S_{n+1}(f)$ for n odd. Parameterizing the odd integers, we have

$$\begin{aligned} S_{2n+1}(f)(t) &= \sum_{k=0}^{n} \frac{2i}{\pi} \left(\frac{e^{i2\pi(2k+1)t}}{2k+1} - \frac{e^{-i2\pi(2k+1)t}}{2k+1} \right) \\ &= \sum_{k=0}^{n} \frac{2i}{\pi} \cdot \frac{2i \sin(2\pi(2k+1)t)}{2k+1} \\ &= -\frac{4}{\pi} \sum_{k=0}^{n} \frac{\sin(2\pi(2k+1)t)}{2k+1}. \end{aligned}$$

Seeking a maximum, we calculate the derivative

$$\begin{aligned}
\frac{d}{dt}S_{2n+1}(f)(t) &= -8\sum_{k=0}^{n}\cos(2\pi(2k+1)t) \\
&= -\frac{4}{\sin(2\pi t)}\sum_{k=0}^{n} 2\sin(2\pi t)\cos(2\pi(2k+1)t) \\
&= -\frac{4}{\sin(2\pi t)}\sum_{k=0}^{n} \sin(2\pi(2k+2)t)-\sin(2\pi(2k)t) \\
&= -\frac{4[\sin(2\pi(2n+2)t)-\sin(2\pi(2\cdot 0)t)]}{\sin(2\pi t)} \\
&= -\frac{4\sin(2\pi(2n+2)t)}{\sin(2\pi t)},
\end{aligned}$$

which vanishes when

$$2(2n+2)t = m \in \mathbb{Z} \qquad \text{and} \quad t \notin \frac{1}{2}\mathbb{Z}.$$

That is,

$$t_m = \frac{m}{2(2n+2)}.$$

The global maximum is achieved for the smallest m such that $t_m > \frac{1}{2}$. Alternatively, it is also achieved for the largest m such that $t_m < 0$. The latter is $m = -1$, and then

$$\begin{aligned}
S_{2n+1}(f)(t_{-1}) &= -\frac{4}{\pi}\sum_{k=0}^{n}\frac{\sin(-\pi\frac{2k+1}{2n+2})}{2k+1} \\
&= 4\sum_{k=0}^{n}\frac{\sin(\pi\frac{2k+1}{2n+2})}{\pi\frac{2k+1}{2n+2}}\cdot\frac{1}{2n+2}.
\end{aligned}$$

In the limit,

$$\begin{aligned}
\lim_{n\to\infty} 4\sum_{k=0}^{n}\frac{\sin(\pi\frac{2k+1}{2n+2})}{\pi\frac{2k+1}{2n+2}}\cdot\frac{1}{2n+2} &= 2\lim_{n\to\infty}\sum_{k=0}^{n}\frac{\sin(\pi\frac{2k+1}{2n+2})}{\pi\frac{2k+1}{2n+2}}\cdot\frac{1}{2n+2} + 2\lim_{n\to\infty}\sum_{k=0}^{n}\frac{\sin(\pi\frac{2k+2}{2n+2})}{\pi\frac{2k+2}{2n+2}}\cdot\frac{1}{2n+2} \\
&= 2\lim_{n\to\infty}\sum_{k=1}^{2n+2}\frac{\sin(\pi\frac{k}{2n+2})}{\pi\frac{k}{2n+2}}\cdot\frac{1}{2n+2} \\
&= 2\int_0^1 \operatorname{sinc}(t)\ dt \approx 1.17898,
\end{aligned}$$

which gives approximately an overshoot of 8.949% of the jump height.

2. Note: it should be assumed that $a > 0$.
Solution using a differential equation. Using integration by parts, we find

$$\begin{aligned}
\hat{f}'(\omega) &= \frac{d}{d\omega}\int e^{-(a-ib)t^2}e^{-i\omega t}\ dt \\
&= \int -ite^{-(a-ib)t^2}e^{-i\omega t}\ dt \\
&= -\int i\frac{e^{-(a-ib)t^2}}{2(a-ib)}(-i\omega)e^{-i\omega t}\ dt \\
&= \frac{-\omega}{2(a-ib)}\hat{f}(\omega).
\end{aligned}$$

Note that no boundary terms appear from integration by parts as $a > 0$ ensures $te^{-(a-ib)t^2}e^{-i\omega t} \to 0$ as $|t| \to \infty$. Hence

$$\begin{aligned}\frac{d}{d\omega}\ \log \hat{f}(\omega) &= \frac{-\omega}{2(a-ib)}\\ \log \hat{f}(\omega) &= \frac{-\omega^2}{4(a-ib)} + C_0\\ \hat{f}(\omega) &= C_1 e^{-\frac{\omega^2}{4(a-ib)}},\end{aligned}$$

where

$$C_1 = \hat{f}(0) = \int f(t)\ dt = \sqrt{\frac{\pi}{a-ib}},$$

Indeed,

$$\begin{aligned}C_1^2 &= \int\int e^{-(a-ib)(x^2+y^2)}\ dxdy\\ &= \int_0^{2\pi}\int_0^{\infty} re^{-(a-ib)r^2}\ drd\theta\\ &= \frac{\pi}{a-ib},\end{aligned}$$

and we understand $\sqrt{\cdot}$ to mean the square root with positive real part.

Solution using contour integrals. Define $\zeta = a - ib$ and $h(z) = e^{-z^2}$, which is analytic. We can write

$$\hat{f}(\omega) = \int h(\sqrt{\zeta}t)e^{-i\omega t}\, dt,$$

where $\sqrt{\zeta}$ is the square root of ζ having positive real part, defined with the branch cut of $(-\infty, 0]$. We complete the square

$$-\zeta t^2 - i\omega t = -\zeta\left(t^2 + 2\frac{i\omega}{2\zeta}t\right) = -\zeta\left(t + \frac{i\omega}{2w}\right)^2 - \frac{1}{4\zeta}\omega^2,$$

so that

$$h(\sqrt{\zeta}t)e^{-i\omega t} = h\left(\sqrt{\zeta}\left(t + \frac{i\omega}{2\zeta}\right)\right)e^{-\frac{1}{4\zeta}\omega^2}.$$

We need to compute

$$\int h\left(\sqrt{\zeta}\left(t + \frac{i\omega}{2\zeta}\right)\right)\ dt.$$

We will use complex-analytical arguments. We need to justify two changes of variables. Since the real part of ζ is positive, we can use the same arguments as in Lecture 1 with the function $g(z) = e^{-\zeta z^2}$ to obtain

$$\int h\left(\sqrt{\zeta}\left(t + \frac{i\omega}{2\zeta}\right)\right)\ dt = \int h(\sqrt{\zeta}t)\, dt.$$

Since $h(\sqrt{\zeta}t) = h(-\sqrt{\zeta}t)$, we only need to compute

$$\int_0^{\infty} h(\sqrt{\zeta}t)\, dt.$$

Consider the contour $\Gamma_R = \Gamma_{1,R} \cup \Gamma_{2,R} \cup \Gamma_{3,R}$ as where

$$\begin{aligned}
\Gamma_{1,R} &= \{z : \mathrm{Im}(z) = 0,\, 0 \le \mathrm{Re}(z) \le R\}, \\
\Gamma_{2,R} &= \{z : |z| = R,\, 0 \le \mathbf{arg}(z) \le \mathbf{arg}(\sqrt{\zeta})\}, \\
\Gamma_{3,R} &= \{z : z = t\sqrt{\zeta},\, 0 \le t \le R\}.
\end{aligned}$$

where in $\Gamma_{2,R}$ we assumed $\arg(\sqrt{\zeta}) \ge 0$. Otherwise the inequality should read $0 \ge \mathbf{arg}(z) \ge \mathbf{arg}(\sqrt{\zeta})$. Note that

$$\left| \int_{\Gamma_{2,R}} h(z)\, dz \right| \le (2\pi R)\, \mathbf{arg}(\sqrt{\zeta})\, e^{-c_0 R^2} \xrightarrow{R \to \infty} 0,$$

for some $c_0 > 0$. By taking limits on the other integrals and using the analyticity of h we deduce

$$0 = \int_\Gamma h(z)\, dz = \int_{\Gamma_1} h(z)\, dz + \int_{\Gamma_3} h(z)\, dz,$$

where $\Gamma = \Gamma_1 \cup \Gamma_3$ are the curves

$$\begin{aligned}
\Gamma_1 &= \{z : \mathbf{Im}(z) = 0,\, 0 \le \mathbf{Re}(z)\}, \\
\Gamma_2 &= \{z : z = t\sqrt{\zeta},\, t \ge 0\}.
\end{aligned}$$

Since

$$\int_{\Gamma_1} h(z)\, dz = \int_0^\infty h(t)\, dt \quad \text{and} \quad \int_{\Gamma_3} h(z)\, dz = -\sqrt{\zeta} \int_0^\infty h(\sqrt{\zeta} t)\, dt,$$

we conclude

$$\int_0^\infty h(\sqrt{\zeta} t)\, dt = \frac{1}{\sqrt{\zeta}} \int_0^\infty h(t)\, dt = \frac{1}{2}\sqrt{\frac{\pi}{\zeta}}.$$

Finally

$$\hat{f}(\omega) = e^{-\frac{1}{4\zeta}\omega^2} \int h\left(\sqrt{\zeta}\left(t + \frac{i\omega}{2\zeta}\right)\right) dt = e^{-\frac{1}{4\zeta}\omega^2} \int h(\sqrt{\zeta} t)\, dt = \sqrt{\frac{\pi}{\zeta}} e^{-\frac{1}{4\zeta}\omega^2} = \sqrt{\frac{\pi}{a - ib}} e^{-\frac{1}{4(a-ib)}\omega^2}.$$

3. (a) Yes, we can reconstruct f. With the samples given, the discretized signal is

$$f_d(t) = \sum_{n \in \mathbb{Z}} f\left(\frac{2n}{3}\right) \delta\left(t - \frac{2n}{3}\right).$$

The aliasing formula tells us

$$\hat{f}_d(\omega) = \frac{3}{2} \sum_{k \in \mathbb{Z}} \hat{f}\left(\omega - 3\pi k\right).$$

Because f is bandlimited on $[-\pi, \pi]$, we have

$$\hat{f}(\omega) = \frac{2}{3} \hat{\varphi}(\omega) \hat{f}_d(\omega).$$

Note that this equality fails if samples are taken at integers (i.e. $T = 1$). The convolution theorem now gives

$$f(t) = \frac{2}{3} \cdot \left(\left(\sum_{n \in \mathbb{Z}} f\left(\frac{2n}{3}\right) \delta_{\frac{2n}{3}} \right) * \varphi \right)(t) = \frac{2}{3} \sum_{n \in \mathbb{Z}} f\left(\frac{2n}{3}\right) \varphi\left(t - \frac{2n}{3}\right), \tag{1}$$

the desired reconstruction.

(b) The obvious con is that more samples are required. The main pro is that φ has a faster decay rate than sinc, and so (1) has a faster rate of convergence than does Shannon interpolation. That is, fewer samples are needed for the interpolation to return an accurate result.

Now, why does φ have a faster decay rate? Well, smooth functions have fast decay in the frequency domain. We saw in Lecture 7 this fact for signals on a periodic domain. The argument for Fourier transforms on $\mathbb{R}$ follows the same observations: If $\psi \in L^1(\mathbb{R})$ is p-times differentiable and tends to 0 at $\pm\infty$, then for $\omega \neq 0$,

$$\hat{\psi}(\omega) = \int \psi(t)e^{-i\omega t}\, dt = \frac{1}{i\omega}\int \psi'(t)e^{-i\omega t}\, dt = \frac{1}{(i\omega)^p}\int \psi^{(p)}(t)e^{-i\omega t}\, dt.$$

Hence

$$|\hat{\psi}(\omega)| \leq \frac{\|\psi^{(p)}\|_1}{|\omega|^p},$$

meaning $\hat{\psi}(\omega) = O(|\omega|^{-p})$ as $|\omega| \to \infty$ (assuming $\psi^{(p)} \in L^1(\mathbb{R})$). For a smooth function, this holds for every p, and so the decay rate in the frequency domain is faster than any polynomial.

Now, strictly speaking this analysis is for Fourier transforms, not for *inverse* Fourier transforms. Nevertheless, it should be clear that the same arguments go through for the inverse transforms of smooth functions. Interestingly, though, we don't even need the analysis for the inverse transform, since the functions of interest are even. For a real-valued even function $\hat{\psi} \in L^1(\mathbb{R})$,

$$\psi(t) = \frac{1}{2\pi}\int \hat{\psi}(\omega)e^{i\omega t}\, d\omega = \frac{1}{2\pi}\int \hat{\psi}(\omega)e^{-i\omega t}\, d\omega = \frac{1}{2\pi}\hat{\hat{\psi}}(t).$$

That is, the inverse Fourier transform becomes just a multiple of the Fourier transform. In this problem, $\hat{\varphi}$ is smooth while the boxcar window is not even continuous. Correspondingly, φ has a fast rate of decay, $O(|t|^{-p})$ for every p, whereas the sinc function does not, $O(|t|^{-1})$.

4. (a) No, since the function is not bandlimited. The Fourier transform of a Gaussian is a Gaussian.

(b) Yes, it is close to bandlimited. Gaussians decay very quickly, and so discretizing in the time domain does not lead to much aliasing in the frequency domain. Shannon interpolation, for instance, would then give a reconstruction with few artifacts from aliasing.

(c) We are graphing

$$\sum_{n\in\mathbb{Z}} f(nT)\mathrm{sinc}\left(\frac{t-nT}{T}\right)$$

for various values of T. As T tends to 0, the approximation to the Gaussian density becomes more accurate because aliasing is being reduced (arbitrarily small, in fact) in the frequency domain. See the first and second plots below.

(d) Answers vary with details of numerical implementation. See the third plot below for an example.

Problem 4

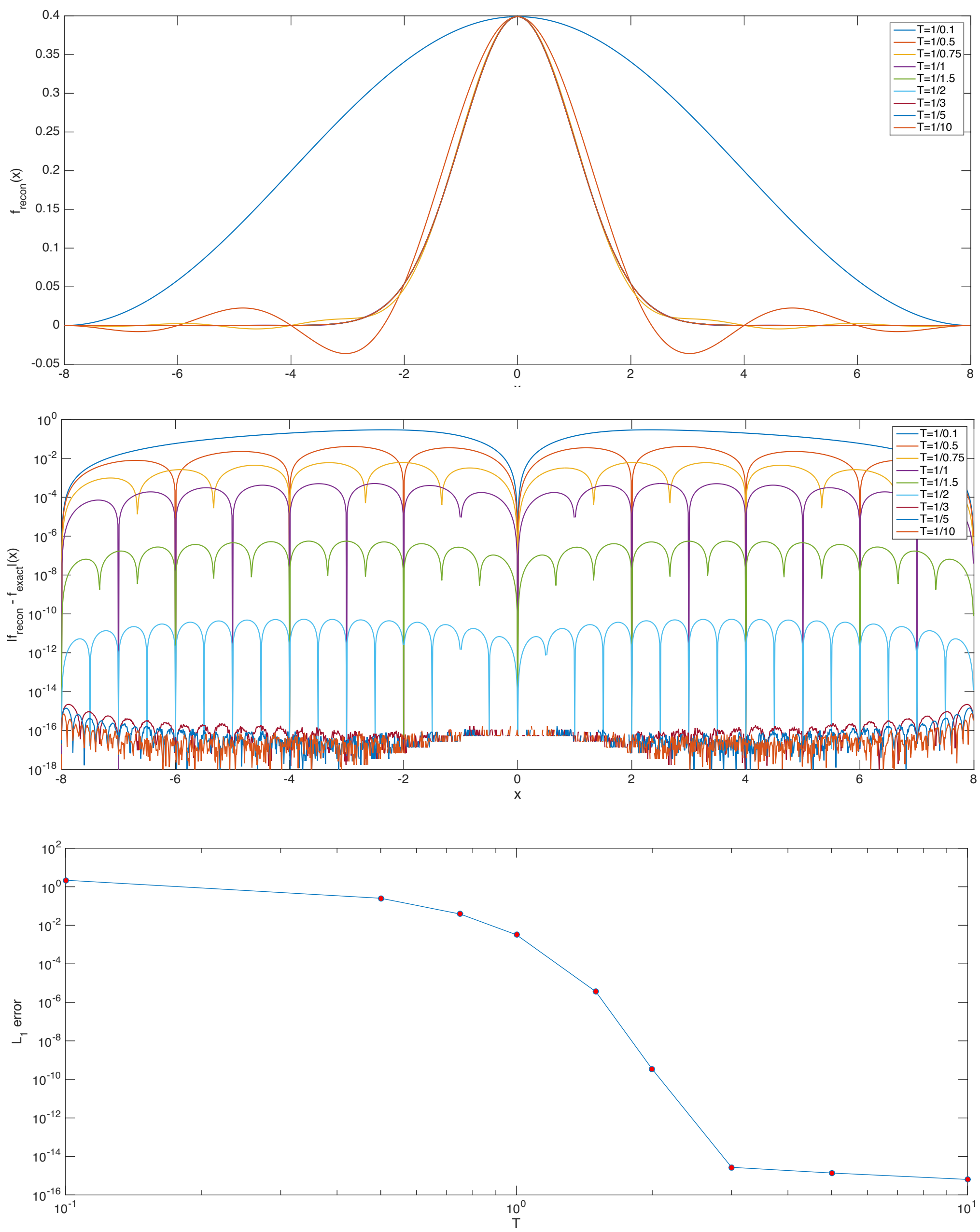


Figure 2: *Graphics due to Aditya Ghate.* The first shows Shannon interpolation for various values of T, the second the corresponding pointwise error, and the third the measured distortion.