Math 262 / CME 372 **Winter 2021**

Homework 1

Due February 2

1. **Gibbs Phenomenon.** Gibbs' phenomenon has to do with how poorly Fourier series converge in the vicinity of a jump or discontinuity of a signal f. This fact was pointed out by Gibbs in a letter to *Nature* (1899). (Actually Gibb's phenomenon was first described by the British mathematician Wilbraham (1848).) The function Gibbs considered was a sawtooth (1). Gibbs was replying to a letter by the physicist Michaelson to *Nature* (1898), in which the latter expressed himself doubtful as to the idea that a real discontinuity (in f) can a replace a sum of continuous curves ($S_n(f)$). Recall that the Fourier series of a periodic function f on the unit interval is

$$f(t) = \sum_{k\in\mathbb{Z}} c_k\, e^{i2\pi kt} \quad \text{with} \quad c_k = \int_0^1 f(t)e^{-i2\pi kt}\, dt.$$

To investigate Gibb's phenomenon, let us look at the function on the unit interval

$$f(t) = \begin{cases} t & 0 \le t < 1/2 \\ t-1 & 1/2 \le t < 1 \end{cases} \tag{1}$$

   (a) Calculate the Fourier coefficients (c_k) of f.

   (b) Let $S_n(f)$ be the partial sum $S_n = \sum_{|k|\le n} c_k e^{i2\pi kt}$. Calculate the approximation error $\|f - S_n(f)\|_{L_2}$ as accurately as possible.

   (c) Consider now
$$f(t) = \begin{cases} -1 & 0 \le t < 1/2 \\ 1 & 1/2 \le t < 1 \end{cases}.$$
   Gibbs observed that in the vicinity of the jump, the partial sums always overshoot the mark by about 9%. Verify this assertion by carefully setting up a numerical experiment.

   (d) Repeat the last question on the sawtooth signal (1).

   (e) *Bonus question.* Can you prove that in (d)
$$\lim_{n\to\infty} \max S_n \approx 1.089.$$

2. *Chirps.* Compute the Fourier transform of the function
$$f(t) = \exp(-(a-ib)t^2).$$
Such a function is called a chirp.

3. *Other sampling theorems?* I know a function $\hat{\varphi}$ with the following properties: $\hat{\varphi}$ is even, equal to 1 for $\omega \in [-\pi, \pi]$, and smoothly decays to zero as shown in Figure 1. Also, imagine I have a signal f bamdlimited to $[-\pi, \pi]$.

   (a) Suppose f is a signal bamdlimited to $[-\pi, \pi]$. Explain how we can reconstruct f from samples $\{f(2n/3)\}_{n\in\mathbb{Z}}$, i.e. from data sampled at a rate 50% higher than the Nyquist rate. [*Hint:* Can you relate $\hat{f}(\omega)$ and $\hat{f}_d(\omega)\hat{\varphi}(\omega)$?)

   (b) Explain why the reconstruction may be attractive from a practical standpoint. What pros and cons can you think of? [*Hint:* Recall that the Fourier transform exchanges smoothness for decay so that $\varphi(t)$ may have an interesting property you might like to discuss.]

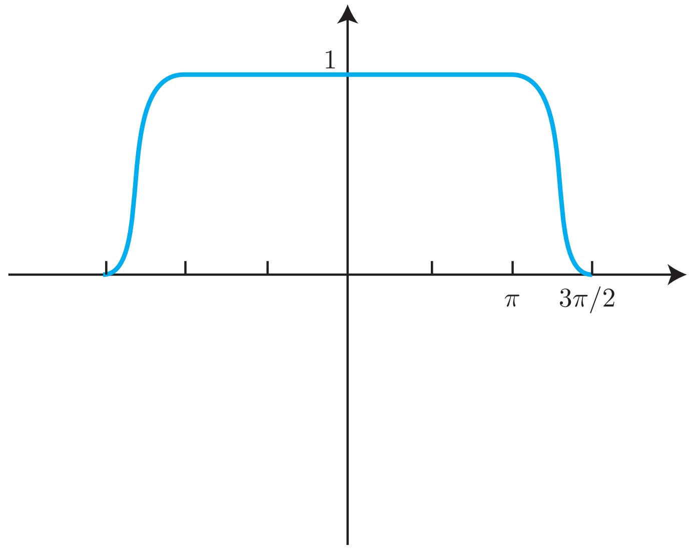

Figure 1: Filter

4. *Numerical interpolation.* In this problem, $f(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$ is the usual Gaussian density function.

   (a) Can this function be reconstructed exactly from a set of samples $\{f(nT)\}_{n\in\mathbb{Z}}$ by means of Shannon's interpolation theorem?

   (b) Can it be reconstructed perhaps not exactly but with nevertheless high precision? Why or why not?

   (c) Display reconstructed signals (via Shannon's formula) for various values of $1/T = 1$, $1/T = 5$, $1/T = 10$ (and perhaps other values of T). What do you see?

   (d) For what value of T do you obtain that the distortion defined here as $\int |f(t) - \tilde{f}(t)|\, dt < 10^{-2}$? 10^{-4}? ($\tilde{f}$ is the reconstruction via Shannon's formula.)

   (e) *Bonus problem.* Same question as in (d) but if you use a filter like that from Figure 1.