Unitary matrices

As we have seen in class, an $n \times n$ unitary matrix $\boldsymbol{U}$ is a matrix obeying

$$\boldsymbol{U}^*\boldsymbol{U} = \boldsymbol{I}. \tag{1}$$

Expressed differently, a unitary matrix is a matrix whose inverse is its adjoint. If the columns of $\boldsymbol{U}$ are the vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$, the (i, j)th entry of $\boldsymbol{U}^*\boldsymbol{U}$ is equal to $\langle \boldsymbol{u}_i, \boldsymbol{u}_j \rangle$ and, therefore, another way to write (1) is

$$\langle \boldsymbol{u}_i, \boldsymbol{u}_j \rangle = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

This says that the family $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\}$ is an orthonormal basis of $\mathbb{C}^n$. In short, the columns of a unitary matrix form an orthobasis.

Now recall that when we write $\boldsymbol{U}\boldsymbol{x} = \boldsymbol{b}$, or equivalently

$$x_1\boldsymbol{u}_1 + \ldots + x_n\boldsymbol{u}_n = \boldsymbol{b}, \tag{2}$$

we can think of $\boldsymbol{x}$ as the coordinates of the vector $\boldsymbol{b}$ in the basis $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\}$. These coordinates are given by

$$\boldsymbol{x} = \boldsymbol{U}^{-1}\boldsymbol{b} = \boldsymbol{U}^*\boldsymbol{b} \quad \Leftrightarrow \quad x_i = \langle \boldsymbol{u}_i, \boldsymbol{b} \rangle, \ i = 1, \ldots, n.$$

Hence, we can rewrite (2) as

$$\langle \boldsymbol{u}_1, \boldsymbol{b} \rangle \boldsymbol{u}_1 + \ldots + \langle \boldsymbol{u}_n, \boldsymbol{b} \rangle \boldsymbol{u}_n = \boldsymbol{b}.$$

Observe that we can rewrite the left-hand side as $(\boldsymbol{u}_1\boldsymbol{u}_1^* + \ldots + \boldsymbol{u}_n\boldsymbol{u}_n^*)\boldsymbol{b}$ so that

$$(\boldsymbol{u}_1\boldsymbol{u}_1^* + \ldots + \boldsymbol{u}_n\boldsymbol{u}_n^*)\boldsymbol{b} = \boldsymbol{b}.$$

Since this holds *for all* $\boldsymbol{b} \in \mathbb{C}^n$, this means nothing else than

$$\boldsymbol{u}_1\boldsymbol{u}_1^* + \ldots + \boldsymbol{u}_n\boldsymbol{u}_n^* = \boldsymbol{I}. \tag{3}$$

In conclusion, whenever $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\}$ is an orthobasis, (3) always holds.

Suppose now that you give me $m \leq n$ orthonormal vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$. Then the orthonormal projection onto the span of these vectors is given by

$$\boldsymbol{P} = \boldsymbol{u}_1\boldsymbol{u}_1^* + \ldots + \boldsymbol{u}_m\boldsymbol{u}_m^*.$$

When $m < n$, this cannot be the identity matrix because the column space of $\boldsymbol{P}$ is $\mathrm{span}(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m)$, which is of dimension $m < n$.