ACM 116 Problem Set 4 Solutions

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Problem 1

Answer (a) is correct.

Suppose I arrive at time t, and the first arrival after t is bus $N_i + 1$ and occurs at time T_{N_i+1}. Let $W_t = T_{N_i+1} - t$, which is the waiting time or forward recurrence time at t. By the memoryless property, it's clear that W_t is exponentially distributed with rate λ, $E[W_t] = \frac{1}{\lambda}$. Also, for fixed t, let

$$V_t = \begin{cases} t - T_{N_i}, & \text{if } N_i > 0 \\ t & \text{if } N_i = 0 \end{cases}$$

be the backward recurrence time at t, the time that has elapsed at t since the last arrival bus prior to t (or t if there have been no arrivals). It's clear that $0 \leq V_t \leq t$, thus V_t cannot be exponentially distributed. Actually, V_t has the cumulative distribution function (figure 1)

$$F(s) = \begin{cases} 0 & s < 0 \\ 1 - e^{-\lambda s} & 0 \leq s < t \\ 1 & s \geq t \end{cases}$$

As $t \to \infty$, F converges to the cumulative distribution function of an exponential distribution. V_t has a distribution function P. Notice that since F has discontinuity at t, P is point mass at t.

$$P(s) = \begin{cases} 0 & s < 0 \\ \lambda e^{-\lambda s} & 0 \leq s < t \\ e^{-\lambda t}\delta(s-t) & s = t \\ 0 & s > t \end{cases}$$

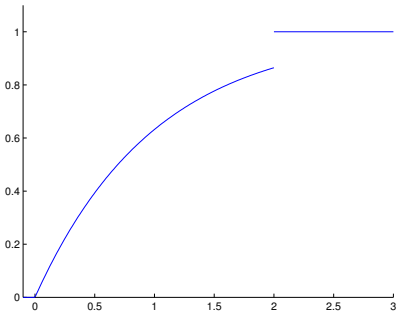


Figure 1: Cumulative Distribution Function F for V_t

We can compute

$$\begin{aligned} E[V_t] &= \int_0^t s\lambda e^{-\lambda s} ds + te^{-\lambda t} \\ &= -te^{-\lambda t} + \frac{1-e^{-\lambda t}}{\lambda} + te^{-\lambda t} \\ &= \frac{1-e^{-\lambda t}}{\lambda} \end{aligned}$$

W_t and V_t are independent for each t by the memoryless property. Therefore we see that W_t and V_t are not symmetric for fixed t, although as $t \to \infty$, $E(W_t) = E(V_t) = \frac{1}{\lambda}$.
This solution is based on pp160, 'Probability' by A F Karr, Springer-Verlag.

Problem 2

The Poisson process in the problem is a process of randomly occurring events in space such that:

i) For any region of volume V the number of events in that region has a Poisson distribution with mean λV.

ii) The number of events in nonoverlapping regions are independent.

Fix the star *alpha* at origin $y = 0$, consider the region $V_1 = \{y \in \mathbb{R}^3 | 0 < |y| < x\}$ and $V_2 = \{y \in \mathbb{R}^3 | x < |R| < x + dx\}$, the probability P that the nearest neighbor to *alpha* belongs to $V2$ is,

$$\begin{aligned} P &= P(N(V_1) = 0)P(N(V_2) \geq 1) \\ &= e^{-\lambda V_1}(\lambda V_2 + o(V_2)) \\ &= 4\lambda\pi x^2 e^{-\frac{4\lambda\pi x^3}{3}} dx \end{aligned}$$

the probability distribution function is given by

$$f_R(x) = 4\lambda\pi x^2 e^{-\frac{4\lambda\pi x^3}{3}},\ x > 0.$$

Problem 3

(a) State i can be defined as: i customers in the station, $m - i$ customers are outside the station and each will enter the system will exponential rate θ, $0 \leq 0 \leq m$.
The balance equation is

state	entering rate	=	leaving rate
0	μP_1	=	$m\theta P_0$
$0 < i < m$	$(m-i+1)P_{i-1} + \mu P_{i+1}$	=	$((m-i)\theta + \mu)P_i$
m	θP_{m-1}	=	μP_m

which can be solved by

$$P_i = \frac{m!}{(m-i)!}\left(\frac{\theta}{\mu}\right)^i P_0$$

and

$$\sum_{i=0}^{m} P_i = 1.$$

(b) Average rate that customer enter the system is

$$\lambda_a = \sum_{i=0}^{m} P_i \theta (m-i) = m\theta - \theta \sum_{i=0}^{m} iP_i = \theta(m-L).$$

The average number of customers in the system L is

$$L = \sum_{i=0}^{m} iP_i.$$

(c) The average time that a customer spends in the station per visit is

$$W = L/\lambda_a = \frac{L}{\theta(m-L)} = \frac{\sum_{i=0}^{m} iP_i}{\theta m - \theta \sum_{i=0}^{m} iP_i}$$

Problem 4

(a) State i can be defined as the state that i customer in the system. The balance equation is,

state	entering rate	=	leaving rate
0	μP_1	=	λP_0
1	$\lambda P_0 + 2\mu P_2$	=	$(\lambda + \mu) P_1$
2	$\lambda P_1 + 2\mu P_3$	=	$(\lambda + 2\mu) P_2$
3	λP_2	=	$2\mu P_3$

(b) Let $\rho = \frac{\lambda}{\mu}$. The solution of the balance equation is given by the long run probabilities,

$$\begin{aligned}
P_0 &= \frac{1}{1 + \rho + \frac{1}{2}\rho^2 + \frac{1}{4}\rho^3} \\
P_1 &= \frac{\rho}{1 + \rho + \frac{1}{2}\rho^2 + \frac{1}{4}\rho^3} \\
P_2 &= \frac{\frac{1}{2}\rho^2}{1 + \rho + \frac{1}{2}\rho^2 + \frac{1}{4}\rho^3} \\
P_3 &= \frac{\frac{1}{4}\rho^3}{1 + \rho + \frac{1}{2}\rho^2 + \frac{1}{4}\rho^3}
\end{aligned}$$

(c) If a customer arrives and finds two others in the system, his waiting time is exponential with rate 2μ and service time is exponential with rate μ. Therefore the expected time he spends in the system is $E(T) = \frac{1}{2\mu} + \frac{1}{\mu} = \frac{3}{2\mu}$.

(d) Customers can enter if less than 3 customers in the system. The proportion of customers entering the system is

$$1 - P_3 = \frac{\rho + \frac{1}{2}\rho^2 + \frac{1}{4}\rho^3}{1 + \rho + \frac{1}{2}\rho^2 + \frac{1}{4}\rho^3}.$$

(e) The rate of customers entering the system is

$$\lambda_a = \lambda(1 - P_3).$$

The average number of customers in the system is

$$\begin{aligned}
L &= P_1 + 2P_2 + 3P_3 \\
&= \frac{\rho + \rho^2 + \frac{3}{4}\rho^3}{1 + \rho + \frac{1}{2}\rho^2 + \frac{1}{4}\rho^3}
\end{aligned}$$

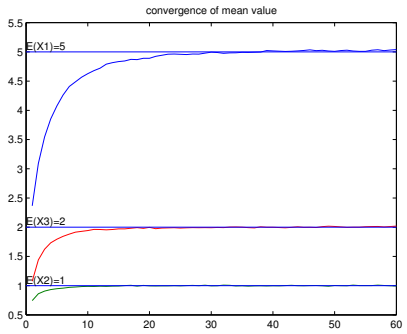


Figure 2: convergence plot

The average time a customer in the system is

$$W = \frac{L}{\lambda_a} = \frac{\rho + \rho^2 + \frac{3}{4}\rho^3}{\lambda(1 + \rho + \frac{1}{2}\rho^2)}$$

Problem 5

(a) By the fact that $\lambda < \min\{\mu_1, \mu_2, \mu_3\}$ and Burke's Theorem, there exist long run probabilities (steady state) and the departure process of each station is also Poisson with rate λ.
Let $\rho = \frac{\lambda}{\mu_3} = 10/15 = 2/3$, the probability of k customers in the station is

$$\rho^k(1-\rho).$$

Suppose c is integer, we have

$$\lim_{t\to\infty} P(X_3(t) > c) = \sum_{c+1}^{\infty} \rho^k(1-\rho) = \rho^{c+1}.$$

The smallest number c for which $\lim_{t\to\infty} P(X_3(t) > c) < 0.01$ is given by $c = \left\lceil \frac{\log(0.01)}{\log \rho} \right\rceil - 1 = 11$, where $\lceil x \rceil$ means the minimum integer larger than x.

(b) See the accompanying `Matlab` file in the Appendix.
In the long run, the averaged time in server i is $L_i = \frac{\lambda}{\mu_i - \lambda}$, thus $L_1 = E(X_1) = 5$, $L_2 = E(X_2) = 1$ and $L_3 = E(X_3) = 2$. We study the convergence by sampling X_i and drawing the averaged X_i at different time points, and compare it with $E(X_i)$. In figure 2, we can see that after about 30 hours the mean value converges visually.

Appendix

`Matlab`-code for Problem 2

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% This code is modified from Mulin Cheng's homework
%
% The function prob5 is for running the numerical
% experiments
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
function prob5
  close all
  clear all

  lambda=10;
  mu1=12; mu2=20; mu3=15;

  % simulate the process up to time T
  T=60;
  N=60;
  % sample time points
  tlist=linspace(T/N,T,N);
  samplesdrawn=100000;

  X1=zeros(samplesdrawn,N); X2=X1; X3=X1;
  X10=zeros(1,N); X20=X10; X30=X10;

  for i=1:samplesdrawn
    [X10,X20,X30]=sample(lambda,mu1,mu2,mu3,T,tlist);
    X1(i,:)=X10;
    X2(i,:)=X20;
    X3(i,:)=X30;
  end

  figure
  hold on
  plot(tlist,mean(X1),tlist,mean(X2),tlist,mean(X3));
  line([0 T],[5 5]);text(0,5,'E(X1)=5')
  line([0 T],[2 2]);text(0,2,'E(X3)=2')
  line([0 T],[1 1]);text(0,1,'E(X2)=1')

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% This function is for the sampling with the method described
% in 5(b)
% X_i number of customers in station i at time list tlist
% lambda arrival rate
% mu_i service rate at station i
% T simulation time
% tlist sample time list
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [X1,X2,X3]=sample(lambda,mu1,mu2,mu3,T,tlist)
%the total number of customers entering the system by time t
  N=poissrnd(lambda*T);
  n=length(tlist);
  X1=zeros(1,n);
  X2=zeros(1,n);
  X3=zeros(1,n);

  % arrival times
  AT0=T*rand(N,1);
  % ordered arrival times
  AT=sort(AT0,'ascend');

  DT1=DepartTime(AT,mu1);
```

```
  DT2=DepartTime(DT1,mu2);
  DT3=DepartTime(DT2,mu3);

  for i=1:n
    t=tlist(i);

    % M is the arrivals at server 1 before time t
    M=find(AT<=t,1,'last');
    if length(M)==0 % if no arrivals before time t
      M=0;
      X1(i)=0;
      X2(i)=0;
      X3(i)=0;
    end

    % departures from server 1 before time t
    temp=find(DT1(1:M)<=t,1,'last');
    if length(temp)==0 % if no departure before time t
      temp=0;
      X1(i)=M;
      X2(i)=0;
      X3(i)=0;
    end
    % arrivals at server 1 - departures from server 1
    X1(i)=M-temp;

    % M is departures from server 1, i.e. arrivals at server 2
    M=temp;
    % departures from server 2
    temp=find(DT2(1:M)<=t,1,'last');
    if length(temp)==0 % if no departure before time t
      temp=0;
      X2(i)=M;
      X3(i)=0;
    end
    % arrivals at server 2 - departures from server 2
    X2(i)=M-temp;

    % M is departures from server 2, i.e. arrivals at server 3
    M=temp;
    % departures from server 3
    temp=find(DT3(1:M)<=t,1,'last');
    if length(temp)==0 % if no departure before time t
      temp=0;
      X3(i)=M;
    end
    % arrivals at server 3 - departures from server 3
    X3(i)=M-temp;
  end

  return

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% This function is for the departure time of customers
```

```
% passing by a single station with service rate mu
% DT departure times
% AT arrival times
% mu service rate
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function DT=DepartTime(AT,mu)
  n=length(AT);
  DT=zeros(n,1);
  % sample the service time
  ST=exprnd(1/mu,n,1);

  DT(1)=AT(1)+ST(1);

  for i=2:n
    DT(i)=AT(i)+max([DT(i-1)-AT(i),0])+ST(i);
  end

  return
```