ACM106a - Homework 3 Solutions

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1. **Chapter 7, Problem 7.5:**

   (a) Note that by equation (7.8) we have:

   $$|r_{jj}| = \|a_j - \sum_{i=1}^{j-1} r_{ij} q_i\|_2$$

   Therefore, $r_{jj} = 0$ iff a_j if a linear combination of $a_1, a_2 \ldots a_{j-1}$. Thus, A has rank n iff all r_{jj} are nonzero.

   (b) Note that 0 is the root of characteristic polynomial of $\hat{R}$ with multiplicity $n-k$. Then $dim(Null(\hat{R})) \geq k$, therefore, $k \leq Rank(\hat{R}) < n$, and, consequently, $k \leq Rank(A) < n$.

2. **Chapter 10, Problem 10.2:**

   (a)
```
function [W,R]=house(A)
  m=size(A,1);
  n=size(A,2);
  W=zeros(m,n);
  for k=1:n
    x=A(k:m,k);
    e1=eye(m-k+1,1);
    if (x(1)==0) coef=1;
       else coef=sign(x(1));
    end;
    W(k:m,k)=coef*norm(x,2)*e1+x;
    W(k:m,k)=W(k:m,k)/norm(W(k:m,k),2);
    A(k:m,k:n)=A(k:m,k:n)-2*W(k:m,k)*((W(k:m,k))'*A(k:m,k:n));
  end;
  R=triu(A,0);
```

   (b)
```
function Q=formQ(W)
m=size(W,1);
Q=zeros(m,m);
for i=1:m
  e=zeros(m,1);
  e(i)=1;
  Q(:,i)=formQx(W,e);
end;

function a=formQx(W,vect)
m=size(W,1);
```

```
n=size(W,2);
for k=n:-1:1
   vect(k:m)=vect(k:m)-2*W(k:m,k)*((W(k:m,k))'*vect(k:m));
end;
a=vect;
```

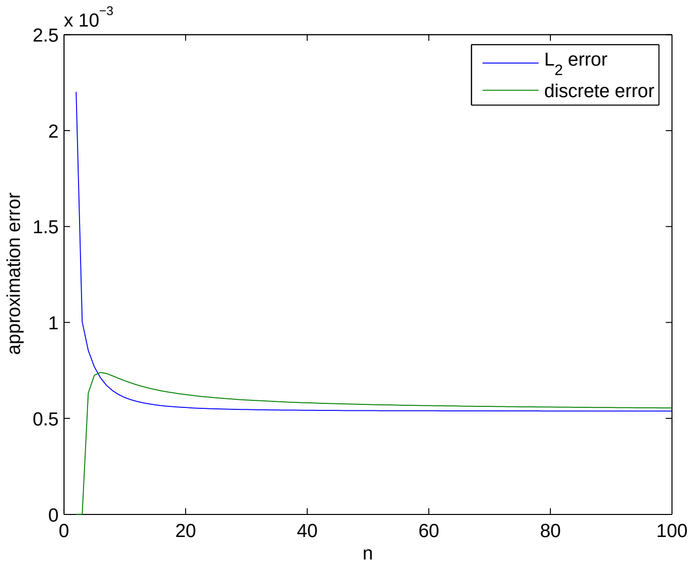
3. **Chapter 11, Problem 11.2(a):**

```
function pr_11_2
 for n=2:100
  %set up the least squares problem
  A=zeros(n,3);
  x=linspace(1,2,n);
  A(:,1)=exp(x');
  A(:,2)=sin(x');
  A(:,3)=gamma(x');
  b=1./x';

  %solve least squares problem (use QR factorization)
  [Q,R]=qr(A);
  sol=R\(Q'*b);

  %compute L_2 error
  error(n)=sqrt(quad(@(z)fcnint(z, sol(1), sol(2), sol(3)),1,2));
  %compute discrete L_2 error
  discrete_error(n)=sqrt((norm(b-sol(1)*A(:,1)-sol(2)*A(:,2)-sol(3)*A(:,3)))^2/n);
 end;
 plot(2:100,error(2:100),2:100,discrete_error(2:100));

 function y=fcnint(z,c1,c2,c3)
  y=(1./z-c1*exp(z)-c2*sin(z)-c3*gamma(z)).^2;
```



Then from the graph we can see that for $n = 10$, for example, the coefficients are: -0.10760253505158, 0.01095054034647, 1.28443893104056. The error of approximation is 6.093666653341175e-004 in continuous case and 6.962798320742533e-004 in discrete case.

4. **Chapter 11, Problem 11.3:**

```
function pr_11_3(m,n)

 %set up the problem
 t=linspace(0,1,m);
 B=vander(t);
 B=fliplr(B);
 A=B(:,1:n);
 b=cos(4*t);

 %method (a)
 xa=(A'*A)\(A'*b');

 %method (b)
 [Q,R]=mgs(A);
 xb=R\(Q'*b');

 %method (c)
 [W,R]=house(A);
 Q=formQ(W);
 xc=R\formQtb(W,b');

 %method (d)
 [Q,R]=qr(A);
 xd=R\(Q'*b');

 %method (e)
 %note that by default matlab uses QR decomposition for the least squares
 %problem
 xe=A\b';

 %method (f)
 [U,S,V]=svd(A);
 w=S\(U'*b');
 xf=V*w;

 %print out the table of solutions
 soln(:,1)=xa;
 soln(:,2)=xb;
 soln(:,3)=xc;
 soln(:,4)=xd;
 soln(:,5)=xe;
 soln(:,6)=xf;
 soln

 %evaluate the polynomial at each point
 ya=polyval(fliplr(xa'),A(:,2));
 yb=polyval(fliplr(xb'),A(:,2));
 yc=polyval(fliplr(xc'),A(:,2));
 yd=polyval(fliplr(xd'),A(:,2));
 ye=polyval(fliplr(xe'),A(:,2));
 yf=polyval(fliplr(xf'),A(:,2));
 plot(t,ya,'b',t,yb,'g',t,yc,'r',t,yd,'c',t,ye,'m',t,yf,'y',t,cos(4*t),'k');
```

```
function [Q,R]=mgs(A)
%QR factorization using modified Gram-Schmidt
n=size(A,2);
R=zeros(n,n);
for i=1:n
  V(:,i)=A(:,i);
end;
for i=1:n
  R(i,i)=norm(V(:,i),2);
  Q(:,i)=V(:,i)/R(i,i);
  for j=(i+1):n
     R(i,j)=Q(:,i)'*V(:,j);
     V(:,j)=V(:,j)-R(i,j)*Q(:,i);
  end;
end;

%implicit calculation of a product Q'b
function a=formQtb(W,vect)
m=size(W,1);
n=size(W,2);
for k=1:n
   vect(k:m)=vect(k:m)-2*W(k:m,k)*((W(k:m,k))'*vect(k:m));
end;
a=vect;
```

The result of running the program for $m = 50$, $n = 12$ (the columns of soln are the coefficients obtained by each method):

```
soln =
 1.00000001060074  0.99999999859093  1.00000000099660  1.00000000099661  1.00000000099661
 -0.00000328567061 0.00000027088201  -0.00000042274291 -0.00000042274316 -0.00000042274332
 -7.99987307544056 -8.00000704985627 -7.99998123568637 -7.99998123568344 -7.99998123567749
 -0.00192050937461 0.00005898516353  -0.00031876323797 -0.00031876324901 -0.00031876333094
 10.68174784539845 10.66655515934153 10.66943079598515 10.66943079595884 10.66943079654419
 -0.06962594129663 -0.00090455925668 -0.01382028843134 -0.01382028805576 -0.01382029054076
 -5.48828876180120 -5.68354616965529 -5.64707562610852 -5.64707562757566 -5.64707562089245
 -0.36704155033020 -0.00876156857361 -0.07531602646623 -0.07531602335914 -0.07531603505147
 2.03929133920078  1.61521325639978  1.69360696581747  1.69360696186545  1.69360697513982
 -0.24909531610432 0.06358015838896  0.00603210719689  0.00603211021495  0.00603210078168
 -0.26758631659985 -0.39818374763656 -0.37424170286096 -0.37424170413863 -0.37424170032592
 0.06875193469410  0.09235164808582  0.08804057597642  0.08804057620719  0.08804057553828

 1.00000000099661
 -0.00000042274317
 -7.99998123568297
 -0.00031876325505
 10.66943079599901
 -0.01382028821519
 -5.64707562717213
 -0.07531602402879
 1.69360696259227
 0.00603210971752
 -0.37424170394376
```

```
0.08804057617387
```

By comparing the solutions obtained all methods, we see that methods (c)-(f) produce 7-8 correct digits, whereas methods (a) and (b) have 0-1 correct digits.

5. **Problem 5:**

   (a) The dimension of the solution space (i.e. the dimension of the null space) is $n - m$.

   (b) **Normal equations approach:**
   Since A has a full rank, AA^T is nonsingular, and x has to be represented by $x = A^T(AA^T)^{-1}b$ (you can convince yourself it is true by noting that for $Ax = b$, $x = A^Ty$ solving for y gives $y = (AA^T)^{-1}b$). Now let's show that x is indeed the minimum 2-norm solution. Let's assume that y is another solution, and define $z = y - x$.
   Then

$$\begin{aligned} \|y\|_2^2 &= \|x+z\|_2^2 = \|x\|_2^2 + \|z\|_2^2 + 2(x^Tz) \\ &= \|x\|_2^2 + \|z\|_2^2 \end{aligned}$$

   since $A(y-x) = Ay - Ax = b - b = 0$.
   Therefore, $\|x\|_2 \leq \|y\|_2$ and x is the minimum-norm solution.
   QR approach:
   Because A has more rows than columns, let's decompose A^T instead of A into QR. We have:

$$Q^TA^T = \begin{pmatrix} R_1 \\ 0 \end{pmatrix}$$

   So the system $Ax = b$ becomes $(R_1^T, 0^T)Q^Tx = b$.
   Let $y = Q^Tx$. Then we have $(R_1^T, 0^T)y = (R_1^T, 0^T)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = b$ or $R_1^Ty_1 = b$.
   Then the minimum norm solution is obtained by setting $y_2 = 0$. Therefore, the minimum norm solution to the system is given by $x = Qy = (Q_1, Q_2)\begin{pmatrix} y_1 \\ 0 \end{pmatrix} = Q_1y_1$.