# ACM 104

Homework Set 5 Solutions

February 21, 2001

**1** Franklin Chapter 4, Problem 4, page 102.
Let $A$ be an $n \times n$ non-Hermitian matrix. Suppose that $A$ has distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. Show that $A^*$ has the eigenvalues $\overline{\lambda}_1, \ldots, \overline{\lambda}_n$. If $u^j$ is an eigenvector of $A$ belonging to $\lambda_j$, and if $v^k$ is an eigenvector of $A$ belonging to $\overline{\lambda}_k$, show that $(u^j, v^k) = 0$ if $j \neq k$, and show that $(u^j, v^j) \neq 0$. This is known as the principle of biorthogonality.

Using theorem 2 on page 77 we are able to say that since $A$ has distinct eigenvalues, it has distinct eigenvectors. Furthermore, there exists a matrix $T$ such that

$$\Lambda = T^{-1}AT$$

where $\Lambda$ is the diagonal matrix

$$\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

or

$$A = T\Lambda T^{-1} \Rightarrow A^* = (T\Lambda T^{-1})^* = (T^{-1})^*\Lambda^* T^*$$

But we know that $(T^{-1})^* = (T^*)^{-1}$ and $\Lambda^* = \overline{\Lambda}$ so $A^* = (T^*)^{-1}\overline{\Lambda}T^*$. Therefore, $A^*$ is similar to the matrix $\overline{\Lambda}$, and we conclude that $A^*$ has $n$ distinct eigenvalues (and $n$ distinct eigenvectors) with

$$\overline{\Lambda} = \begin{pmatrix} \overline{\lambda}_1 & & 0 \\ & \ddots & \\ 0 & & \overline{\lambda}_n \end{pmatrix}$$

Now consider

$$(Au^j, v^k) = (\lambda_j u^j, v^k) = \lambda_j (u^j, v^k)$$

and

$$(Au^j, v^k) = (u^j, A^* v^k) = (u^j, \overline{\lambda}_k v^k) = \lambda_k (u^j, v^k)$$

Therefore,

$$\lambda_j (u^j, v^k) = \lambda_k (u^j, v^k)$$

Since the eigenvalues are distinct and non-zero, we must have that for $j \neq k$, $(u^j, v^k) = 0$ for this equality to hold.

Now suppose that $(u^j, v^j) = 0$. We know that the $u^j$'s are independent and there are $n$ of them in an $n$ dimensional space so they form a basis. Therefore, for any $v^k$ we may write

$$v^k = \sum_{i=1}^{n} a_i u^i$$

with not all $a_i = 0$. Now consider

$$(v^j, u^j) = 0 \Rightarrow \left( \sum_{i=1}^{n} a_i u^i, u^j \right) = 0 \Rightarrow \sum_{i=1}^{n} a_i (u^i, u^j) = 0$$

for any $j$. Since this is true we conclude that the $u^i$'s must be dependent since $a_i \neq 0$ for all $i$. This is a contradiction, therefore $(u^j, v^k) \neq 0$ when $j = k$.

**2** Franklin Chapter 4, Problem 4, page 109.
If a Hermitian matrix $H$ is positive definite, show that $H = P^2$, where $P$ is also positive definite.

The diagonalization of $H$ is such that $U^* H U = \Lambda$,

$$\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

Since $H$ is positive definite we know that the diagonal elements of $\Lambda$ are all positive. We can therefore define

$$A = \begin{pmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_n} \end{pmatrix}$$

If we now also define $P = U^* A U$ then we have

$$P^2 = U^* A U U^* A U = U^* A A U = U^* \Lambda U = H,$$

i.e. $P^2 = H$. Note also that $P$ has only positive eigenvalues and so is positive definite.

**3** Franklin Chapter 4, Problem 5, page 109.
If a Hermitian matrix $H$ is positive definite, show that it satisfies the generalized Schwarz inequality,

$$|(Hx, y)| \leq (Hx, x)^{\frac{1}{2}} (Hy, y)^{\frac{1}{2}}$$

and the generalized triangle-inequality,

$$(H(x+y), x+y)^{\frac{1}{2}} \leq (Hx, x)^{\frac{1}{2}} + (Hy, y)^{\frac{1}{2}}$$

Since $H$ is Hermitian, there exists a unitary matrix $U$ such that $U^* H U = \Lambda$ with

$$\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

But $H$ is positive definite so all $\lambda_i > 0$. Therefore, we can write $\Lambda = A^2$ where

$$A = \begin{pmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_n} \end{pmatrix}$$

Since $A$ is diagonal and real, note that $A = A^*$ and we also have $U^*HU = A^2 = A^*A = AA^*$ so $H = UAA^*U^* = UA(UA)^*$. Now,

$$\begin{aligned} |(Hx,y)| &= |(UA(UA)^*x,y)| = |((UA)^*x,(UA)^*y)| \\ &\leq ||(UA)^*x||.||(UA)^*y|| \\ &= ((UA)^*x,(UA)^*x)^{\frac{1}{2}}((UA)^*y,(UA)^*y)^{\frac{1}{2}} \\ &= (UA(UA)^*x,x)^{\frac{1}{2}}(UA(UA)^*y,y)^{\frac{1}{2}} \\ &= (Hx,x)^{\frac{1}{2}}(Hy,y)^{\frac{1}{2}} \end{aligned}$$

therefore

$$|(Hx,y)| \leq (Hx,x)^{\frac{1}{2}}(Hy,y)^{\frac{1}{2}}$$

Now note that

$$\begin{aligned} (H(x+y),x+y) &= (Hx+Hy,x+y) = (Hx,x)+(Hy,x)+(Hx,y)+(Hy,y) \\ &\leq (Hx,x)+2|(Hy,x)|+(Hy,y) \\ &= (Hx,x)+2(Hx,x)^{\frac{1}{2}}(Hy,y)^{\frac{1}{2}}+(Hy,y) \\ &= ((Hx,x)^{\frac{1}{2}}+(Hy,y)^{\frac{1}{2}})^2 \end{aligned}$$

therefore

$$(H(x+y),x+y)^{\frac{1}{2}} \leq (Hx,x)^{\frac{1}{2}}+(Hy,y)^{\frac{1}{2}}$$

**4** Franklin Chapter 4, Problem 4, page 115.
If $U^*AU = T$ is a unitary triangularization, show that

$$\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2 = \sum_{i\leq j}|t_{ij}|^2$$

We have

$$AA^* = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}\begin{pmatrix} \overline{a}_{11} & \dots & \overline{a}_{n1} \\ \vdots & \ddots & \vdots \\ \overline{a}_{1n} & \dots & a_{nn} \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{n}|a_{1k}|^2 & & \ddots \\ & \ddots & \\ \ddots & & \sum_{k=1}^{n}|a_{nk}|^2 \end{pmatrix}$$

Similarly,

$$TT^* = \begin{pmatrix} \sum_{k=1}^{n}|t_{1k}|^2 & & \ddots \\ & \ddots & \\ \ddots & & \sum_{k=1}^{n}|t_{nk}|^2 \end{pmatrix}$$

Then note that

$$\mathrm{tr}(AA^*) = \sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2, \mathrm{tr}(TT^*) = \sum_{i=1}^{n}\sum_{j=1}^{n}|t_{ij}|^2 = \sum_{i\leq j}|t_{ij}|^2$$

So, in order to prove the above result we need only show that $\mathrm{tr}(AA^*) = \mathrm{tr}(TT^*)$. Now, since $A = UTU^*$ we have

$$AA^* = UTU^*(UTU^*)^* = UTU^*U^*TU = UTU^*UT^*U^* = UTT^*U^*,$$

i.e. $AA^*$ is similar to $TT^*$. Since the trace is invariant under a change of coordinates we are done.

**5** Franklin Chapter 4, Problem 5, page 115.
Consider a system of differential equations $dx/dt = Ax$, where $A$ is not similar to a diagonal matrix. Let $U^*AU = T$ be a unitary triangularization of $A$. Show how the change of variable $x = Uy$ allows the system to be resolved recursively for $y_n$, then $y_{n-1}, \ldots$, and finally $y_1$.

We have $\frac{dx}{dt} = Ax$ replacing $x$ with $Uy$ we get

$$\frac{d}{dt}Uy = AUy$$

now multiply both sides by $U^*$ so that

$$U^*\frac{d}{dt}Uy = U^*AUy \Rightarrow \frac{d}{dt}y = Ty$$

that is,

$$\begin{pmatrix} y_1' \\ \vdots \\ y_n' \end{pmatrix} = \begin{pmatrix} \lambda_1 & & \ldots \\ & \ddots & \ldots \\ 0 & & \lambda_n \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

or

$$\begin{aligned} y_n' &= \lambda_n y_n \\ y_{n-1}' &= \lambda_{n-1}y_{n-1} + t_{n-1,n}y_n \\ &\ldots \end{aligned}$$

So the first equation will allow us to solve for $y^n$. Substituting this into the second equation will allow us to solve for $y_{n-1}$. Continuing in this manner, we have $y$ solved.

**6** Franklin Chapter 4, Problem 4, page 120.
Let $N^*N = NN^*$. Show that there is positive-semidefinite Hermitian matrix, $P$, and a unitary matrix, $V$, such that $N = PV = VP$. (Use the factorization, $N = U\Lambda U^*$. If $NN^* = U\Lambda\Lambda^*U^* = P^2$, how can $P$ be defined?)

Note that

$$\Lambda\Lambda^* = \begin{pmatrix} |\lambda_1|^2 & & 0 \\ & \ddots & \\ 0 & & |\lambda_n|^2 \end{pmatrix}$$

so define

$$P = U \begin{pmatrix} |\lambda_1|^2 & & 0 \\ & \ddots & \\ 0 & & |\lambda_n|^2 \end{pmatrix} U^*$$

$P$ is clearly Hermitian, and since the eigenvalues of $P$ are non-negative, $P$ is semi-definite. So now we want to find $V$ unitary such that $N = PV$. If $N$ is nonsingular, all eigenvalues will be nonzero so we can write $V = P^{-1}N$. $V$ is unitary since

$$VV^* = P^{-1}NN^*P^{-1} = P^{-1}U\Lambda\Lambda^*U^*P^{-1} = P^{-1}P^2P^{-1} = I$$

Now suppose that $N$ is singular. We want to find $V$ such that $VV^* = I$ and $PV = U\Lambda U^*$. Assume $V = U^*DU$ where $D$ is diagonal. So

$$PV = U \begin{pmatrix} |\lambda_1|^2 & & 0 \\ & \ddots & \\ 0 & & |\lambda_n|^2 \end{pmatrix} U^*UDU^*$$

so we want

$$\begin{pmatrix} |\lambda_1|^2 & & 0 \\ & \ddots & \\ 0 & & |\lambda_n|^2 \end{pmatrix} D = \Lambda$$

Looking at it term by term, we want $|\lambda_i|\mu_i = \lambda_i$ so $\mu_i = \frac{\lambda_i}{|\lambda_i|}$. If $|\lambda_i| = 0$ let $\mu_i = 1$. Let

$$D = \begin{pmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_2 \end{pmatrix}$$

where

$$\mu_i = \begin{cases} \frac{\lambda_i}{|\lambda_i|}, \text{if } |\lambda_i| \neq 0 \\ 1, \text{if } |\lambda_i| = 0 \end{cases}$$

so $V = U^*DU$. Note that $VV^* = UDD^*U = I$ so $U$ is unitary and by our definition $VP = U$.

**7** A skew Hermitian matrix is a matrix that satisfies $K^* = -K$.

1. Show that $K = U\Lambda U^*$, with $\Lambda$ diagonal and for a unitary $U$.
2. Show that the eigenvalues are imaginary and the eigenvectors orthogonal.
3. Show that $K + I$ is invertible.
4. Show that $(I - K)(I + K)^{-1}$ is a unitary matrix.
5. Show that $e^{Kt}$ is a unitary matrix.

Consider the matrix $iK$. Then

$$(iK)^* = (i)^*K^* = -i(-K) = iK$$

so $iK$ is hermitian.

1. Since $iK$ is hermitian, by theorem 2 on page 100 there exist a unitary matrix $U$ and a diagonal matrix $\Lambda$, so that

$$iK = U\Lambda U^* \Rightarrow K = U(-i)\Lambda U^* \tag{1}$$

2. The matrix $K$ has the same eigenvectors as $iK$ and its eigenvalues are the eigenvalues of $iK$, multiplied by $-i$. Since $iK$ is hermitian, $K$ has orthogonal eigenvectors and its eigenvalues are imaginary.

3. To find the eigenvalues of $K + I$ we just add 1 to the eigenvalues of $K$. If $v$ is an eigenvector of $K$ and $\lambda$ is the corresponding eigenvalue:

$$\begin{aligned} Kv &= \lambda v \\ (K + I)v &= (\lambda + 1)v \end{aligned}$$

Since the eigenvalues of $K$ are imaginary, the eigenvalues of $K + I$ are all greater than 1 in absolute value, therefore nonzero and $K + I$ is invertible.

4. Consider

$$\begin{aligned}((I-K)(I+K)^{-1})^*(I-K)(I+K)^{-1} &= ((I+K)^{-1})^*(I-K)^*(I-K)(I+K)^{-1} \\ &= ((I+K)^{-1})^*(I+K)(I-K)(I+K)^{-1}\end{aligned}$$

Note that $(I+K)$ and $(I-K)$ commute for any matrix $K$, so

$$\begin{aligned}((I-K)(I+K)^{-1})^*(I-K)(I+K)^{-1} &= ((I+K)^{-1})^*(I-K)(I+K)(I+K)^{-1} \\ &= ((I+K)^{-1})^*(I-K) \\ &= ((I+K)^{-1})^*(I+K)^* \\ &= ((I+K)(I+K)^{-1})^* \\ &= I^* \\ &= I\end{aligned}$$

So $(I-K)(I+K)^{-1}$ is unitary.

5. Let $\lambda_1, \ldots, \lambda_n$ denote the imaginary eigenvalues of $K$:

$$\begin{aligned}e^{Kt} &= \sum_{j=1}^{\infty} \frac{K^j t^j}{j!} \\ &= \sum_{j=1}^{\infty} \frac{(U\Lambda U^*)^j t^j}{j!} \\ &= U\left(\sum_{j=1}^{\infty} \frac{\Lambda^j t^j}{j!}\right) U^* \\ &= U\begin{pmatrix} \sum_{j=1}^{\infty} \frac{\lambda_1^j t^j}{j!} & & \\ & \ddots & \\ & & \sum_{j=1}^{\infty} \frac{\lambda_n^j t^j}{j!} \end{pmatrix} U^* \\ &= U\begin{pmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{pmatrix} U^*\end{aligned}$$

Since the eigenvalues $\lambda_i$ are imaginary, $e^{\lambda_i}$ is a complex number with magnitude 1. Using the last expression it is obvious that $(e^{Kt})^* e^{Kt} = I$ so $e^{Kt}$ is unitary.