ACM 104

Homework Set 2 Solutions

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1 Franklin Chapter 1, Problem 7, page 20.
Let A be an $n \times n$ triangular matrix, with $a_{ij} = 0$ for $j < i$. Assuming all $a_{ii} \neq 0$, prove that A^{-1} is also a triangular matrix.

We will do the proof only in the case of A being upper triangular. The other case is similar. Since $a_{ii} \neq 0$ and A is upper triangular the inverse A^{-1} makes sense. For each n-dimensional vector b:

$$Ax = b \Rightarrow \begin{pmatrix} a_{11} & \dots & \dots & \dots \\ 0 & a_{22} & \dots & \dots \\ 0 & 0 & \dots & \dots \\ 0 & 0 & 0 & a_{nn} \end{pmatrix} x = b \Rightarrow \left\{ \begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & = & b_1 \\ & & a_{22}x_2 & + & \dots & = & b_2 \\ & & & \dots & \dots & = & \cdot \\ & & & & a_{nn}x_n & = & b_n \end{array} \right\}$$

We can solve these equations for b by starting from the last one and going upwards. It is obvious that x_n depends only on b_n, x_{n-1} depends only on $b_{n-1}, b_n, \dots$, x_k depends only on $b_n, \dots, b_k$. For

$$x = A^{-1}b$$

this implies that A^{-1} is upper triangular.

2 Franklin Chapter 2, Problem 3, page 37.
Let x and y be in the linear space $E^n(R)$ of column-vectors with n real components. These vectors are said to be orthogonal if $x_1y_1 + \cdots + x_ny_n = 0$. Let $x^1, \dots, x^k$ be nonzero vectors in $E^n(R)$. Let every two of these vectors be orthogonal.
Show that the vectors $x^1, \dots, x^k$ are linearly independent.

Let $a_1, \cdots + a_k$ be such that

$$a_1x^1 + \cdots + a_qx^q + \ldots + a_kx^k = 0$$

Multiply by the transpose of x^q on the left:

$$a_1(x^q)^Tx^1 + \ldots + a_q(x^q)^Tx^q + a_k(x^q)^T = 0$$

Since $(x^q)^Tx^i = x_1^qx_1^i + \cdots + x_n^qx_n^i = 0$,when $i \neq q$ we have

$$a_q(x^q)^Tx^q = 0$$

Since x^q are nonzero vectors this implies that $a_q = 0$,for each q, so the vectors are linearly independent.

3 Franklin Chapter 2, Problem 7, page 37.
Consider the vector space of $m \times n$ matrices with complex components over the field of complex numbers. What is a basis for this space? What is the dimension?

It is easy to check that the matrices $A_{i,j}$, $1 \leq i \leq m, 1 \leq j \leq n$ that have zero entries but for the ith row and jth column where it is one, i.e.

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & 1 \end{pmatrix} \tag{1}$$

is the basis for our space (they are trivially linearly independent and span the space). Therefore the dimension of this space is mn.

4 Franklin Chapter 2, Problem 4, page 43.
Prove that $\text{rank}(A+B) \leq \text{rank}(A) + \text{rank}(B)$.

The rank of a matrix can be defined as the dimension of the space that its column vectors span. Let $(a^1, ..., a^n)$ denote the column vectors of A, $(b^1, ..., b^n)$ denote the column vectors of B and $(c^1, ..., c^n)$ denote the column vectors of $A+B$. The column vectors c^i can be written as a linear combination of a^i, b^i so the dimension of the space spanned by them cannot be greater than the sum of the dimensions of the spaces spanned by A and B.

5 Franklin Chapter 2, Problem 5, page 43.
Prove that $\text{rank}(AB) \leq \text{rank}(B)$. Prove that $\text{rank}(AB) \leq \text{rank}(A)$, and hence, $\text{rank}(AB) \leq min\{\text{rank}(A), \text{rank}(B)\}$.

To multiply A by B on the right means that every column of the result AB is a linear combination of the columns of A. Using the above definition for the rank of a matrix we get the desired result, that $\text{rank}(AB) \leq \text{rank}(B)$. The dimension of the space spanned by linear combinations of the column vectors of A, is certainly not bigger than the dimension of the space spanned by the column vectors of A.

Equivalently, the rank of a matrix is the dimension of the space spanned by its row vectors. The rows of AB arise by a linear combination of the rows of B, as the relation $AB = (A^T B^T)^T$ implies. The dimension of the space spanned by linear combinations of rows of B cannot be larger than the dimension of the space spanned by the rows of B, so $\text{rank}(AB) \leq \text{rank}(A)$.

6 Find the dimension of the space of $n \times n$ symmetric matrices, as well as a basis.

We will assume that the matrices are defined over a field E (E could be $\mathbb{R}$ or $\mathbb{C}$ or anythiing else). A symmetric matrix is known if one only knows the elements on and above the diagonakl, say. the dimension of the space.
A basis for the space of $n \times n$ symmetric matrices may be obtained as follows: for $1 \leq k \leq \ell \leq n$, define the matrix $A_{k,\ell}$ by

$$A_{k,\ell}(i,j) = \begin{cases} 1 & (i,j) = (k,\ell) \text{ or } (j,i) = (k,\ell) \\ 0 & otherwise \end{cases}$$

That is:

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & 1 \end{pmatrix},$$

and

$$\begin{pmatrix} 0 & 1 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ 1 & 0 & \dots & 0 \end{pmatrix}, \dots .$$

It is easy to check that these matrices are linearly independent and span the space. There are $n(n+1)/2$ of them which is then the dimension of the space.

7 If A is an $n \times n$ matrix such that $A^2 = A$ and rank$(A) = n$ prove that A must be the identity.

The rank can also be defined as the size of the largest nonzero determinant of a matrix. If rank$(A) = n$, then $det(A) \neq 0$ so A is invertible. Multiplying the given relation with A^{-1} on the left, we get $A = I$.

8 What is the rank of the $n \times n$ matrix with every entry equal to one? How about the checkboard matrix, with $a_{ij} = 0$ when $i + j$ is even and $a_{ij} = 1$ when $i + j$ is odd?

The dimension of the space spanned by the column vectors of an $n \times n$ matrix with every entry equal to one, is obviously one since there all column vectors are identical, namely, $(1, 1, \dots, 1)$. So its rank is one. Or we can notice that all 2×2 determinants of the matrix are zero, so again its rank is one.

Similarly, the dimension of the space spanned by the column vectors of the checkboard matrix is obviously two since there is only two linearly independent column vectors: $(1, 0, 1, \dots)$and$(0, 1, 0, \dots)$. So its rank is one. Or we can notice that all 3×3 determinants of the matrix are zero, so again its rank is two.