ACM 104

Homework Set 1 Solutions

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1. Franklin Chapter 1, Problem 2, page 9.
   Let A be of the form the form

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ 0 & 0 & a_{33} & a_{34} & a_{35} \\ 0 & 0 & a_{43} & a_{44} & a_{45} \\ 0 & 0 & a_{53} & a_{54} & a_{55} \end{pmatrix} = \begin{pmatrix} A_1 & \dots \\ 0 & A_2 \end{pmatrix}$$

   Show that

$$\det A = \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \det A_1 \det A_2$$

   In the definition of the determinant as a sum $\det A = \sum_{j_1,\dots,j_5} s(j_1,\dots,j_5) a_{1j_1} \dots a_{5j_5}$ the terms containing elements a_{ij} with $i > 2$ and $j < 3$ are zero. In any of the permutations $(j_1, j_2, j_3, j_4, j_5)$ of that sum that give a non-zero term the last three elements j_3, j_4, j_5 must have values 3,4 or 5. This implies that the first two elements j_1, j_2 have values 1 or 2.

   The permutation$(j_1, j_2, j_3, j_4, j_5)$ decouples into the composition of two permutations (j_1, j_2) and (j_3, j_4, j_5). Therefore we can write

$$\begin{aligned} \det A &= \sum_{j_1,\dots,j_5} s(j_1,\dots,j_5) a_{1j_1} \dots a_{5j_5} \\ &= \sum_{j_1,j_2\text{: all permutations of 1,2}} \quad \sum_{j_3,j_4,j_5\text{: all permutations of 3,4,5}} s(j_1,\dots,j_5) a_{1j_1} \dots a_{5j_5} \end{aligned}$$

   Now it is clear that $s(j_1,\dots,j_5)$ is equal to $s(j_1,j_2)s(j_3,j_4,j_5)$ (think of the number of transpositions) and therefore we have

$$\begin{aligned} \det \mathrm{A} &= \sum_{\text{all permutations of 1,2}} \quad \sum_{\text{all permutations of 3,4,5}} s(j_1,j_2)s(j_3,j_4,j_5) a_{1j_1} \dots a_{5j_5} \\ &= \sum_{\text{all permutations of 1,2}} s(j_1,j_2) a_{1j_1} a_{2j_2} \quad \sum_{\text{all permutations of 3,4,5}} s(j_3,j_4,j_5) a_{3j_3} a_{4j_4} a_{5j_5} \\ &= \det A_1 \det A_2 \end{aligned}$$

   which is what needed to be shown.

2. Franklin Chapter 1, Problem 3, page 9.
   Generalize the result of problem 1. Let A be a block triangular matrix

$$A = \begin{pmatrix} A_1 & \cdot & \dots & \cdot \\ 0 & A_2 & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & A_p \end{pmatrix}$$

Show that

$$\det A = \det A_1 \det A_2 \ldots \det A_p$$

We prove the result by induction. The result of problem 1 is easy to generalize when A_1 and A_2 are of any size. Let A be $n \times n$ and A_1 be $m \times m$. We proceed as before.

$$\begin{aligned} \det A &= \sum_{j_1,\ldots,j_n} s(j_1,\ldots,j_n) a_{1j_1} \ldots a_{nj_n} \\ &= \sum_{j_1,\ldots,j_m} \sum_{j_{m+1},\ldots,j_n} s(j_1,\ldots,j_n) a_{1j_1} \ldots a_{nj_n} \end{aligned}$$

where the sum over $j_1,\ldots,j_m$ extends over all permutations of $1,\ldots,m$ and similarly for $j_{m+1},\ldots,j_n$ (permulations of $m+1,\ldots,n$). Then

$$\begin{aligned} \det A &= \sum_{j_1,\ldots,j_m} \sum_{j_{m+1},\ldots,j_n} s(j_1,\ldots,j_m) s(j_{m+1},\ldots,j_n) a_{1j_1} \ldots a_{nj_n} \\ &= \sum_{j_1,\ldots,j_m} s(j_1,\ldots,j_m) a_{1j_1} \ldots a_{mj_m} \sum_{j_{m+1},\ldots,j_n} s(j_{m+1},\ldots,j_n) a_{m+1j_{m+1}} \ldots a_{nj_n} \\ &= \det A_1 \det A_2 \end{aligned}$$

We have proved the identity for $p = 2$. Now suppose it holds for $p = q$, that is suppose

$$\det A = \det A_1 \det A_2 \ldots \det A_q$$

The block triangular matrix A' consisting of $q+1$ blocks can be written as

$$A' = \begin{pmatrix} A_1 & \cdot & \ldots & \cdot \\ 0 & A_2 & \ldots & \cdot \\ \cdot & \cdot & \ldots & \cdot \\ 0 & 0 & \ldots & A_{q+1} \end{pmatrix} = \begin{pmatrix} A & \ldots \\ 0 & A_{q+1} \end{pmatrix}$$

and then $\det A' = \det A \det A_{q+1} = \det A_1 \ldots \det A_{q+1}$,so by induction the identity holds for any $p \in N$.

3. Franklin Chapter 1, Problem 4, page 9.
Is it possible to arrange the letters of the alphabet $a, b, \ldots, z$ in reverse order $z, y, \ldots, a$ by exactly 100 successive interchanges of pairs of letters?

Since there are 26 letters in the english alphabet in takes 13 interchanges to place them in reverse order. If we call that permutation j, then its sign is $s(j) = (-1)^{13} = -1$. Suppose now that it is possible to reverse the order of the letters with 100 interchanges of pairs of letters. This would imply that $s(j) = (-1)^{100} = 1$ which is a contradiction.

It is, therefore, impossible.

4. Franklin Chapter 1, Problem 4, page 13.
Give a simple expression for

$$A = \begin{vmatrix} 1 & a & a^2 & a^4 \\ 1 & b & b^2 & b^4 \\ 1 & c & c^2 & c^4 \\ 1 & d & d^2 & d^4 \end{vmatrix}$$

We note that A as a function of a is a fourth order polynomial with roots b, c and d and that the coefficient of a^3 vanishes. The general form for such a polynomial is $A = C_1(a-b)(a-c)(a-d)(a-C_2)$. From this last expression by setting the coefficient of a^3 to zero we get

$$C_1(-C_2 - d - c - b) = 0 \tag{1}$$

The constant C_1 is the coefficient of the fourth power of A(a). From the given determinant it must be equal to:

$$C_1 = -\begin{vmatrix} 1 & b & b^2 \\ 1 & c & c^2 \\ 1 & d & d^2 \end{vmatrix}$$

This is a second order polynomial in b, with roots c and d. For this polynomial the coefficient of b_2 is $-(d-c)$ so

$$C_1 = -(d-c)(b-c)(b-d) \tag{2}$$

Using (1) and (2) we find the determinant to be

$$\begin{aligned} A &= -(d-c)(b-c)(b-d)(a-b)(a-c)(a-d)(a-(-d-c-b)) \\ A &= (c-d)(b-c)(b-d)(a-b)(a-c)(a-d)(a+b+c+d) \end{aligned}$$

Note that in the expression for A interchanging a and b, say, produces an extra minus sign which is expected from the form of the determinant.

5. Franklin Chapter 1, Problem 5, page 20.
If $\det A \neq 0$, show that $(A^T)^{-1} = (A^{-1})^T$.

Since $\det A \neq 0$ we can write

$$A^{-1} = \frac{1}{(\det A)}(\text{ cof } A)^T$$

Taking the transpose of both sides

$$(A^{-1})^T = \frac{1}{(\det A)} \text{ cof } A \tag{3}$$

The fact that $\det A \neq 0$ implies $\det A^T \neq 0$ and we can write

$$(A^T)^{-1} = \frac{1}{(\det A)}(\text{ cof } A^T)^T$$

Since $\text{cof } A^T = (\text{ cof } A)^T$ the last relation becomes

$$(A^T)^{-1} = \frac{1}{(\det A)} \text{ cof } A \tag{4}$$

(3) and (4) give the desired equality.

6. Franklin Chapter 1, Problem 4, page 23.

Assume that $m < n$. Let A be an $m \times n$ matrix, and let B be an $n \times m$ matrix. Let $\alpha(k_1, \ldots, k_m)$ be the determinant formed from columns $k_1, \ldots, k_m$ of A. Let $\beta(k_1, \ldots, k_m)$ be the determinant formed from rows $k_1, \ldots, k_m$ of B. Using the method in the proof of Theorem 1, show that

$$\det(AB) = \sum_{k_1 < , \ldots, < k_m} \alpha(k_1, \ldots, k_m)\beta(k_1, \ldots, k_m)$$

Let $(AB)_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$ denote the (i, j) element of AB. Then

$$\begin{aligned}
\det(AB) &= \sum_{j_1, \ldots, j_m} s(j_1, \ldots, j_m)(AB)_{1j_1} \ldots (AB)_{mj_m} \\
&= \sum_{j_1, \ldots, j_m} s(j_1, \ldots, j_m) \left(\sum_{k_1=1}^{n} a_{1k_1}b_{k_1j_1} \right) \ldots \left(\sum_{k_m=1}^{n} a_{mk_m}b_{k_mj_m} \right) \\
&= \sum_{k_1=1}^{n} \ldots \sum_{k_m=1}^{n} a_{1k_1} \ldots a_{mk_m} \left(\sum_{j_1, \ldots, j_m} s(j_1, \ldots, j_m)b_{k_1j_1} \ldots b_{k_mj_m} \right)
\end{aligned}$$

The last sum is a row expansion of the determinant of the matrix formed by rows $k_1, \ldots, k_m$ of B, which we called $\beta(k_1, \ldots, k_m)$. If any two rows are the same then that term of the sum will be zero so we can write

$$\det(AB) = \sum_{k_1 \neq k_2 \neq \ldots \neq k_m \in (1, \ldots, n)} a_{1k_1} \ldots a_{mk_m}\beta(k_1, \ldots, k_m).$$

Now $\beta(k_1, \ldots, k_m)$ is a determinant. We can permute the rows, that is the arguments in the parenthesis and put them in increasing order. The permutation that does this is $(k_1, \ldots, k_m)$ and its sign will appear in front of the term. We have

$$\det(AB) = \sum_{k_1 < k_2 < \ldots < k_m \in (1, \ldots, n)} a_{1k_1} \ldots a_{mk_m} s(k_1, \ldots, k_m)\beta(k_1, \ldots, k_m)$$

Using the definition of α this can be rewritten as

$$\det(AB) = \sum_{k_1 < k_2 < \ldots < k_m \in (1, \ldots, n)} \alpha(k_1, \ldots, k_m)\beta(k_1, \ldots, k_m)$$

and this completes the proof.

7. Suppose that every row of A adds up to zero. What is the determinant of A?

It is zero. We can add the second, ..., nth column to the first, and the value of the determinant will not change. Then the elements of the first column will be the sums of each row which are zero. The new determinant has a zero column so it is zero.

8. If the entries of A and A^{-1} are all integers, how do you know that both determinants are 1 or -1?

Taking the determinant of both sides in the equation $AA^{-1} = I$ yields

$$\det A \det A^{-1} = 1 \tag{5}$$

If both A, A^{-1} have integer entries then both determinants are integers since the determinants is the sum of products of integers (and therefore is integral). If the product of two integers is one then those integeres must be 1 or -1.